Mathematical modeling of the risk reinsurance process

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ABSTRACT

The study proposes mathematical models for assessing financial risks and their management processes, which allow making optimal decisions to reduce the risks of financial activities of economic entities. For this, an analysis of financial risks was carried out, taking into account the type of economic entity at risk, the specifics of its activities in the financial market; the main methods of financial risk management are classified, and the financial instruments corresponding to them are identified. In this study, the dynamic properties of financial flows arising from the simulation of artificial financial instruments and the influence of the error in the estimates of the parameters of the mathematical model on the results of financial risk management were investigated.

Key words: Mathematical models, risk assessment, risk management, financial risks, risk.

INTRODUCTION

Insurance is a mechanism for the economic protection of property and human life from loss or damage resulting from undesirable incidents, such as fire, accident, death, disability, etc., subject to payment proportional to the perceived risk (Falin, 2003).

Interest in the theory of life insurance is developing along with the development of the insurance market - an important part of a free market economy. Actuarial analysis, in particular, is becoming an integral aspect of the activities of major insurance companies and banks. Insurance as a system for protecting the property interests of citizens, organizations and the state is a necessary element of modern society.

Reinsurance is a system of economic relations in accordance with which the insurer, accepting risks for insurance, transfers part of the responsibility for them on agreed terms to other insurers in order to create a balanced insurance portfolio and ensure the financial stability of insurance operations.

Reinsurance is insurance by one insurer (reinsurer) on the conditions specified by the contract of the risk of fulfillment of all or part of its obligations to the insured by another insurer (reinsurer).

Insurance protection of business entities and the population is currently of great importance, since it is necessary to ensure the continuity of social production, which depends on various types of unforeseen events, to provide certain guarantees for the social protection of the population, etc. Insurance is always associated with a certain risk of loss of insurance funds. Therefore, consideration and research of models of short-term insurance and reinsurance of risks is an urgent task.

The aim of the work is mathematical modeling of the risk reinsurance process. The tasks of the work are mathematical modeling:

- premium values in the individual risk model;
- the size of the insurance portfolio in the individual risk model;
- income of the insurance company in case of reinsurance of risks;
- own retention limit for reinsurance risks.

Individual risk model

In actuarial mathematics, life insurance models are conditionally divided into two large groups, depending on
whether or not the income from investing collected premiums is taken into account. If not, then they talk about short-term insurance (short-term insurance); usually, an interval of 1 year is considered as such a "short" interval. If so, then we are talking about long-term insurance (long-term insurance). Of course, this division is conditional and, in addition, long-term insurance is associated with a number of other circumstances, for example, underwriting (Bühlmann, 1996).

The simplest type of life insurance is as follows:

The insured pays the \( P \) AZN to the insurance company. (This amount is called the insurance premium (premium); the insured may be the insured himself or another person (for example, his employer).

In turn, the insurance company undertakes to pay the person in whose favor the contract is concluded the sum insured (sum assured) of \( b \) AZN. in the event of the death of the insured within a year for the reasons listed in the contract (and does not pay anything if he does not die within a year or dies for a reason that is not covered by the contract).

The sum insured is often taken as equal to 1 or 1000. This means that the premium is expressed as a fraction of the sum insured or per 1000 sum insured, respectively.

The value of the insurance payment (benefit), of course, is much larger than the insurance premium, and finding the "correct" ratio between them is one of the most important tasks of actuarial mathematics.

The question of how much an insurance company should charge for taking on a particular risk is extremely complex. When solving it, a large number of heterogeneous factors are taken into account: the probability of an insured event, its expected magnitude and possible fluctuations, connection with other risks that have already been accepted by the company, the company’s organizational costs for doing business, the ratio between supply and demand for this type of risk in the insurance market, services, etc. However, the main principle is usually the equivalence of the financial obligations of the insurance company and the insured (Rotar, 1994).

Consider the simplest insurance scheme. The insurance fee is paid in full at the time of conclusion of the contract, the obligation of the insured is expressed in the payment of a premium \( P \). The obligation of the company is to pay the sum insured if an insured event occurs. Thus, the monetary equivalent of the insurer's obligations, \( X \), is a random variable:

In its simplest form, the principle of equivalence of obligations is expressed by the equality \( p = MX \), that is, the expected amount of loss is assigned as a payment for insurance. This premium is called the net premium.

Bought for a fixed premium \( P \) AZN, insurance policy, the insured relieved the beneficiary of the risk of financial losses associated with the uncertainty of the moment of death of the insured. However, the risk itself has not disappeared; taken over by the insurance company (Gerber, 1979).

Therefore, equality \( P = MX \) does not really express the equivalence of the obligations of the insured and the insurer. Although on average both the insurer and the insured pay the same amount, the insurance company has the risk that, due to random circumstances, it may have to pay a much larger amount than \( MX \). The insured has no such risk. Therefore, it would be fair that the payment for insurance should include some premium \( \ell \), which would serve as the equivalent of an accident affecting the company. This allowance is called the insurance (or protective) allowance (or load) (security loading), and \( \theta = \ell / MX \), relative insurance premium (relative security loading). The value of the protective allowance is determined such that the probability that the company will have losses on a certain portfolio of contracts ("go broke") is sufficiently small.

It should be noted that the real insurance premium (gross premium or office premium) is more than the loaded net premium (often several times). The difference between them allows the insurance company to cover administrative expenses, provide income, etc (Daykin, 1994).

The exact calculation of the protective allowance can be made within the framework of risk theory. The simplest model of the functioning of an insurance company, designed to calculate the probability of ruin, is the model of individual risk. It is based on the following simplifying assumptions:

a) a fixed, relatively short period of time is analyzed (so that inflation can be neglected and income from asset investment is not taken into account), usually one year;

b) the number of insurance contracts is \( N \) fixed and not random;

c) the premium is paid in full at the beginning of the analyzed period; there are no receipts during this period;

d) each individual insurance contract is observed and the statistical properties of the individual losses associated with it are known \( X_i \).

It is usually assumed that random variables are \( X_1, \ldots, X_N \) independent in the individual risk model (in particular, catastrophes are excluded when insured events occur simultaneously under several contracts).

Within this model, "ruin" is determined by the total losses in the portfolio \( S = X_1 + \ldots + X_N \). If these total payments are greater than the company’s assets intended for payments on this block of business, \( u \), then the company will not be able to meet all of its obligations (without raising additional funds); in this case one speaks of "ruin" (Embrechts, 1993).
So, the probability of the "ruin" of the company is equal to:
\[ R = P(X_1 + \ldots + X_n \geq u). \]

In other words, the probability of "ruin" is an additional function of the distribution of the value of the company's total losses over the considered period of time.

Since the total payouts \( S \) are the sum of independent random variables, the distribution of a random variable \( S \) can be calculated using classical theorems and methods of probability theory.

First of all is the use of convolutions. Recall that if \( X_1 \) and \( X_2 \) are two independent non-negative random variables with distribution functions \( F_1(x) \) and \( F_2(x) \), respectively, then the distribution function of their sum \( X_1 + X_2 \) can be calculated by the formula (Aivazyan et al., 1983):
\[ F(x) = \int_0^x F_1(x-y)dF_2(y). \]

By applying this formula several times, you can calculate the distribution function of the sum of any number of terms. If random variables \( X_1 \) and \( X_2 \) are continuous, then one usually works with densities \( f_1(x) \) and \( f_2(x) \). The sum density can be calculated using the formula (Aivazyan and Mkhitaryan, 1998):
\[ f(x) = \int_0^x f_1(x-y)f_2(y)dy. \]

If random variables \( X_1 \) and \( X_2 \) are integer, then instead of distribution functions, one usually works with distributions:
\[ p_1(n) = P(X_1 = n), p_2(n) = P(X_2 = n). \]

The distribution of the amount \( p(n) = P(X_1 + X_2 = n) \) can be determined by the formula:
\[ p(n) = \sum_{k=0}^{n} p_1(k) \cdot p_2(n-k). \]

Calculating the ruin probability is often simplified by using generating functions and/or Laplace transforms (Anderson, 1976).

Typically, the number of insured in the insurance company is very large. Therefore, the calculation of the probability of ruin involves the calculation of the distribution function of the sum of a large number of terms. In this case, an accurate direct numerical calculation can lead to problems associated with low probabilities. However, a circumstance that hinders accurate calculation opens up the possibility of a quick and simple approximate calculation. This is due to the fact that as the \( N \) probability grows, it \( P(X_1 + \ldots + X_N < x) \) often has a certain limit (usually it needs to \( X \) change in a certain way along with \( N \), which can be taken as an approximate value of this probability. The accuracy of such aproximations is usually high and satisfies practical needs. The main one is the normal (Gaussian) approximation.

The Gaussian approximation is based on the central limit theorem of probability theory. In its simplest formulation, this theorem looks like this: if the random variables are \( X_1, \ldots, X_N \) independent and equally distributed with mean \( a \) and variance \( \sigma^2 \), then for \( N \to \infty \) the distribution function of the centered and normalized sum
\[ S_N = X_1 + \ldots + X_N - N \cdot a \]
\[ \sigma \sqrt{N} \]
has a limit equal to
\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt. \]

There are numerous generalizations of the central limit theorem to cases where the terms \( X_1 \) have different distributions, are dependent, and so on. We restrict ourselves to the assertion that if the number of terms is large (usually enough to \( N \) be on the order of several tens), and the terms are not very small and not very heterogeneous, then the Gaussian approximation for the probability (Anderson, 1976):
\[ P \left( \frac{S_N - MS_N}{\sqrt{DS_N}} < x \right) \]

Of course, this statement is very vague, but the classical central limit theorem without exact error estimates does not give a clear indication of the scope. The standard Gaussian distribution function \( \Phi(x) \) has been studied in detail in probability theory. There are detailed tables for both the distribution function \( \Phi(x) \) itself and the density (Balabanov, 1996):
Table 1: Function values $1 - \Phi(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1 - \Phi(x)$</th>
<th>$x$</th>
<th>$1 - \Phi(x)$</th>
<th>$x$</th>
<th>$1 - \Phi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>15.87%</td>
<td>2.0</td>
<td>2.28%</td>
<td>3.0</td>
<td>0.14%</td>
</tr>
<tr>
<td>1.1</td>
<td>13.57%</td>
<td>2.1</td>
<td>1.79%</td>
<td>3.1</td>
<td>0.069%</td>
</tr>
<tr>
<td>1.2</td>
<td>11.51%</td>
<td>2.2</td>
<td>1.39%</td>
<td>3.2</td>
<td>0.048%</td>
</tr>
<tr>
<td>1.3</td>
<td>9.68%</td>
<td>2.3</td>
<td>1.07%</td>
<td>3.3</td>
<td>0.034%</td>
</tr>
<tr>
<td>1.4</td>
<td>8.08%</td>
<td>2.4</td>
<td>0.82%</td>
<td>3.4</td>
<td>0.020%</td>
</tr>
<tr>
<td>1.5</td>
<td>6.68%</td>
<td>2.5</td>
<td>0.62%</td>
<td>3.5</td>
<td>0.023%</td>
</tr>
<tr>
<td>1.6</td>
<td>5.48%</td>
<td>2.6</td>
<td>0.47%</td>
<td>3.6</td>
<td>0.011%</td>
</tr>
<tr>
<td>1.7</td>
<td>4.46%</td>
<td>2.7</td>
<td>0.35%</td>
<td>3.7</td>
<td>0.007%</td>
</tr>
<tr>
<td>1.8</td>
<td>3.59%</td>
<td>2.8</td>
<td>0.26%</td>
<td>3.8</td>
<td>0.007%</td>
</tr>
<tr>
<td>1.9</td>
<td>2.87%</td>
<td>2.9</td>
<td>0.19%</td>
<td>3.9</td>
<td>0.005%</td>
</tr>
</tbody>
</table>

Table 2: Quantile values $X_\alpha$.

<table>
<thead>
<tr>
<th>$1 - \alpha$</th>
<th>$X_\alpha$</th>
<th>$1 - \alpha$</th>
<th>$X_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1%</td>
<td>3.090</td>
<td>3%</td>
<td>1.881</td>
</tr>
<tr>
<td>0.5%</td>
<td>2.576</td>
<td>4%</td>
<td>1.751</td>
</tr>
<tr>
<td>one%</td>
<td>2.326</td>
<td>5%</td>
<td>1.645</td>
</tr>
<tr>
<td>2%</td>
<td>2.054</td>
<td>ten%</td>
<td>1.282</td>
</tr>
</tbody>
</table>

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \]

Some values $1 - \Phi(x)$ are given in Table 1.

It is also useful to have a table of quantiles $X_\alpha$ (quantile $X_\alpha$ is defined as the root of the equation $\Phi(x) = \alpha$) corresponding to a sufficiently small ruin probability $1 - \alpha$, they are also shown in Table 2.

Individuals and legal entities conclude an insurance contract with insurance companies in order to get rid of financial losses associated with the uncertainty of the occurrence of certain random events. Prior to the conclusion of the insurance contract, the client had some risk that could lead to accidental losses. After the conclusion of the insurance contract, the client got rid of this risk. In other words, the client makes small deterministic expenses in order to get rid of random losses, which, although unlikely, can be disastrously large for him. However, the risk itself did not disappear - it was taken over by the insurance company. Another thing is that, having a large portfolio of contracts, the insurance company provides itself with an extremely low probability of ruin. However, very large claims are possible, which will lead to the ruin of the company. From this point, the insurance company finds itself in the same situation in which its customers were originally (before the conclusion of insurance contracts) - there is a risk of financial losses associated with the uncertainty of filing very large claims (Balabanov, 1995).

To solve this problem, insurance companies resort to a means - insuring their risk in another company. This type of insurance is called reinsurance.

A company that directly enters into insurance contracts and wants to reinsure part of its risk is called a transfer company, and a company that insures the original insurance company is called a reinsurance company.

Suppose that the transmission company pays all claims on its own up to a certain limit of manats, and for claims exceeding $r$, pays the amount $r$ on its own and sues the reinsurance company for the remaining amount.

If this rule applies to each individual claim, then this type of reinsurance is called excess loss reinsurance. The parameter $r$ is called the retention limit. If this rule is applied to a general claim for a certain period, then this type of reinsurance is called reinsurance that stops losses. The parameter $r'$ in this case is called the franchise.

The reinsurer takes over the risk from the transfer company for a fee. In essence, for the reinsurance company, the operation looks like ordinary insurance. Therefore, the reinsurance fee is set on the same principles as premiums for conventional insurance, i.e. risk reinsurance fee is equal to $(1 + 0^+ \cdot Mh(X))$, where $Mh(X)$ is the expected claim against the reinsurance company, $0^+$, relative premium set by the reinsurance company (Balabushkin, 1996).
We will consider reinsurance contracts only from the point of view of the transfer company. Therefore, we will assume that the relative insurance premium set by the reinsurance company is fixed. The main problem will be in the choice of the reinsurance contract and, above all, in the choice of the main numerical parameter of the contract - the retention limit, which is optimal from the point of view of the transmission company.

Determining the premium in the individual risk model

It is assumed that the company insured \( N \) a person with a probability of death within a year \( q \). The company pays the amount \( b \) in the event of the death of the insured during the year and does not pay anything if this person lives until the end of the year.

The current tasks are (Berzon et al., 1998):

- determination of the total premium,
- determination of the total net premium,
- determination of the total protective allowance sufficient to ensure the probability of ruin of the insurance company of the order of \( R \)%.

To simplify calculations, the value of the sum insured is taken as a unit of measurement of monetary amounts.

In this case, payments under the \( i \)-th contract \( X_i \) take two values: 0 and 1 with probabilities \( 1-q \) and \( q \) respectively. Therefore, the average value and dispersion of payments under one contract will be equal to (Body and Merton, 2000):

\[
MX_i = (1-q) \cdot 0 + q \cdot 1 = q,
\]
\[
MX_i^2 = (1-q) \cdot 0^2 + q \cdot 1^2 = q,
\]
\[
DX_i = MX_i^2 - (MX_i)^2 = q - q^2.
\]

For the average value and variance of total payments \( S = X_1 + \ldots + X_N \), the following is performed:

\[
MS = N \cdot MX_i,
\]
\[
DS = N \cdot DX_i.
\]

Using the Gaussian approximation of the centered and normalized value of total payments, the probability of not ruining the company is presented in the following form (Box and Jenkins, 1974):

\[
P(S < u) = P \left( \frac{S - MS}{\sqrt{DS}} < \frac{u - MS}{\sqrt{DS}} \right) = \Phi \left( \frac{u - MS}{\sqrt{DS}} \right).
\]

According to the formulation of the model, it is required that the ruin probability be no more than \( R \)% For this, the value \( \frac{u - MS}{\sqrt{DS}} \) must be equal to \( x_{(100-R)\%} \), i.e. \( u = x_{(100-R)\%} \cdot \sqrt{DS} + MS \) (the amount insured) or in absolute terms \( u \cdot b \) - the desired total premium (Bocharov, 1993).

The total net premium is found as \( MS \cdot b \), and the total protective allowance is \( x_{(100-R)\%} \cdot \sqrt{DS} \cdot b \).

Determining the size of the insurance portfolio in the individual risk model

Consider the problem of determining the volume of the insurance portfolio on the example of the following individual risk model.

The insurance company offers life insurance contracts for one year. Information regarding the coating structure is given in Table 3.

The relative protective allowance is \( \theta \)%.

Determining the number of contracts necessary to ensure the probability of ruin of the order of \( R \)%.

Let \( N \) - the total number of contracts sold, \( X_i \) - payments under the \( i \)-th contract, \( S = X_1 + \ldots + X_N \) - total payments for the entire portfolio, \( \theta \) - relative protective premium. Then the premium for one contract is equal to (Brigham and Gapensky, 1997):

\[
p = \left( 1 + \frac{\theta}{100} \right) \cdot MX_i.
\]

By condition \( P(S < N \cdot p) = 1 - \frac{R}{100} \). On the other side (Brillinger, 1980):

\[
P(S < N \cdot p) = P \left( \frac{S - MS}{\sqrt{DS}} < \frac{N \cdot p - MS}{\sqrt{DS}} \right) = \Phi \left( \frac{N \cdot p - MS}{\sqrt{DS}} \right)
\]

So

\[
\frac{N \cdot p - MS}{\sqrt{DS}} = x_{(100-R)\%},
\]

\[
\frac{N \cdot (1 + \frac{\theta}{100}) \cdot MX_i - N \cdot MX_i}{\sqrt{N \cdot DX_i}} = x_{(100-R)\%},
\]

\[
\frac{N \cdot \theta}{100} \cdot MX_i = x_{(100-R)\%}.
\]
Hence, for the desired number of contracts, we obtain:

\[ N \approx \frac{x^2}{\left( \frac{\theta}{100} \right)^2 \cdot (MX_i)^2} \]

Reinsurance of risks and analysis of income of an insurance company

The policyholder buys a group insurance contract for a group of N people. The insurer assigns a protective premium \( \theta \)\% and enters into a reinsurance contract for excessive individual losses with a deductible limit \( r \) for each risk. The relative protective premium used by the reinsurer is \( \theta^* \)\%.

At the end of the term of the contract, the insurer calculates the balance of income and expenses. Revenues include premium and expenses consist of insurance claims paid (excluding reinsurer’s share), reinsurance fees and administration costs of \( s \)\% of premium (Brillinger, 1980).

The value of the expected income of the insurer at the end of the contract term is determined if the distribution of individual losses is given in Table 4.

Let \( X_i \) - the amount of payments to the \( i \)-th insured (Table 4 contains the distribution of these random variables), \( X_i^r = \min(X_i, r) \) - the share of the insurer, \( X_i^{\text{r}} = \max(X_i, -r, 0) \) - the share of the reinsurer in the insurance indigination to the \( i \)-th insured.

The expected losses of the reinsurer for one insured person are equal to \( MX^* \).

Accordingly, the total expected losses of the reinsurer are equal to \( N \cdot MX^* \). So, there is a fee for reinsurance protection \( (1 + \theta^*) \cdot N \cdot MX^* \).

Let \( S^r = X_1^r + \ldots + X_N^r \) the share of the insurer in the total losses. Let’s find the distribution of this random variable. For this, its generating function is calculated (Burenin, 1995):

\[ M_{X^r} = (M_{X^*})^N. \]

The coefficients at powers of \( z \) give the required distribution.

Since the total premium under insurance contracts is equal to \( (1 + \theta) \cdot N \cdot MX \), the reinsurance coverage fee

\[ (1 + \theta^*) \cdot N \cdot MX^* \]

is \( (1 + \theta) \cdot N \cdot MX \cdot s \) equal to

\[ D = (1 + \theta) \cdot N \cdot MX^* - (1 + \theta^*) \cdot N \cdot MX^* - (1 + \theta) \cdot N \cdot MX \cdot s - S'. \]

The distribution of the random variable \( D \) is obtained from the distribution of the random variable \( S' \). The average expected income of the insurer will be equal to \( MD \).

Determining the own retention limit for reinsurance risks

The company concludes \( N \) similar life insurance contracts for a period of 1 year. The structure of insurance coverage is shown in Table 5.

The company sets the insurance fee based on the probability of ruin \( R \)\%.

The insurance company intends to conclude an excessive loss reinsurance contract with a retention limit \( r \) (\( a \leq r \leq b \)) (Van Horn, 1996).

The reinsurance company sets a relative premium equal to \( \theta^* \)\%.

Determine the value of the own retention limit \( r \), which would minimize the probability that additional funds will need to be raised to pay off the portfolio under consideration (ruin probability).

Let \( X_i \) - the amount of payments to the \( i \)-th insured (Table 5 contains the distribution of these random variables). The expected value of payments under one contract is \( MX_i \), and the variance is \( DX_i \).

Let \( S = X_1 + X_2 + \ldots + X_N \) the total losses of the insurer. Then the expected loss of the insurer under all contracts is \( MS = N \cdot MX \), and the variance is \( DS = N \cdot DX \).

Using the Gaussian approximation of the centered and normalized value of total payments, the probability of not ruining the company is presented in the following form:

\[ P(S < u) = \Phi \left( \frac{u - MS}{\sqrt{DS}} \right) \]

According to the formulation of the model, it is required that the ruin probability be no more than \( R \)\%. For this, the value \( \frac{u - MS}{\sqrt{DS}} \) must be equal to \( X_{(100-R)/\%} \), that is,
Table 5: Structure of insurance coverage.

<table>
<thead>
<tr>
<th>Probability</th>
<th>Sum insured</th>
</tr>
</thead>
<tbody>
<tr>
<td>Death from natural causes</td>
<td>p</td>
</tr>
<tr>
<td>Death by accident</td>
<td>q</td>
</tr>
</tbody>
</table>

\[ u = x_{(100-R\%)} \cdot \sqrt{DS} + MS, \]
where \( u \) is the fund of the insurance company.

On the other hand, the company’s fund is equal to:

\[ u = MS + \Theta \cdot MS \]

where \( \Theta \) is the insurer’s protection margin.

Then we get that:

\[ x_{(100-R\%)} \cdot \sqrt{DS} + MS = MS + \Theta \cdot MS. \]

Expressing \( \Theta \), we get:

\[ \Theta = \frac{x_{(100-R\%)} \cdot \sqrt{DS}}{MS}. \]

Then the fund of the insurance company is equal to:

\[ p = MS + \frac{x_{(100-R\%)} \cdot \sqrt{DS}}{MS} \cdot MS = MS \left( 1 + \frac{x_{(100-R\%)} \cdot \sqrt{DS}}{MS} \right) \]

Suppose now that the company decides to reinsure claims exceeding \( r \) manats (\( a \leq r \leq b \)) in the reinsurance company. In this case, the \( X_i' = \min(X_i, r) \) payment to the insurance company under one contract (Table 6 contains the distribution of these random variables).

Expected losses after reinsurance are equal \( MX_i' \) for one contract and \( MS' = N \cdot MX_i' \) for the entire portfolio (where \( S' = X_1' + X_2' + \ldots + X_n' \) is the total losses of the insurer after reinsurance). The dispersion of costs is equal \( DX_i' \) for one contract and \( DS' = N \cdot DX_i' \) for the entire portfolio (Vasiliev and Letchikov, 2004).

For a reinsurance company, the average value of the payment under one contract is:

\[ MX_i'' = MX_i' - DX_i'. \]

Therefore, the reinsurance fee for one contract is equal to \( MX_i'' + MX_i' \cdot \Theta' = MX_i'' \cdot (1 + \Theta')\). Then the total reinsurance fee is \( N \cdot MX_i'' \cdot (1 + \Theta')\).

After reinsurance, the premium collected by the company will decrease from \( u \) to:

\[ u' = u - N \cdot MX_i'' \cdot (1 + \Theta'). \]

For the probability \( R' \) that the total payments of the insurance company, \( S' \), is greater than the company's assets, \( u' \), using the Gaussian approximation, we have:

\[ R' = P(S' > u') = P \left( \frac{S' - MS'}{\sqrt{DS'}} > \frac{u' - MS'}{\sqrt{DS'}} \right) \approx 1 - \Phi \left( \frac{u' - MS'}{\sqrt{DS'}} \right) \]

Thus, to minimize the ruin probability \( R' \), you need to choose the parameter \( r \) in such a way that the function \( \frac{u' - MS'}{\sqrt{DS'}} \) takes the \( \sqrt{DS'} \) largest value.

**MATHEMATICAL MODELING OF INDIVIDUAL RISK AND RISK REINSURANCE**

**Modeling the premium value in the individual risk model**

It is assumed that the company insured \( N = 3000 \) a person with a probability of death within a year \( q = 0.03 \). The company pays the amount \( b = 25000 \) in the event of the death of the insured during the year and does not pay anything if this person lives until the end of the year. Defined (Vince, 2001):

the amount of the total premium, the amount of the total net premium, the value of the total protective allowances sufficient to ensure the probability of ruin of the insurance company of the order of \( R = 5\% \).

We accept the value of the sum insured as a unit of measurement of monetary amounts. In this case, payments under the \( i \) th contract \( X_i \) take two values: 0 and 1 with probabilities \( 1 - q \) and \( q \) respectively. So

\[ MX_i = (1 - q) \cdot 0 + q \cdot 1 = q = 0.03, \]

\[ MX_2 = (1 - q) \cdot \cdot q \cdot 1^2 = q = 0.03, \]

\[ DX_i = MX_i^2 - (MX_i)^2 = q - q^2 = 0.03 - 0.03^2 = 0.0291. \]
For the average value and variance of total payments
\[ S = X_1 + \ldots + X_N, \]
the following is performed:
\[ MS = N \cdot MX_i = 3000 \cdot 0.03 = 90, \]
\[ DS = N \cdot DX_i = 3000 \cdot 0.0291 = 87.3. \]

The probability of a company not going bankrupt is presented as follows:
\[ P(S < u) = \Phi \left( \frac{u - MS}{\sqrt{DS}} \right) = \Phi \left( \frac{u - 90}{\sqrt{87.3}} \right). \]

The condition requires that the probability of ruin be no more than 5%. For this, the value \( \frac{u - 90}{\sqrt{87.3}} \) must be equal to \( x_{95} \), that is, \( u = 105.37 \cdot 250000 = 26342492.54 \) the desired total premium.

The total net premium is \( MS \cdot b = 22500000 \), and the total protective allowance is \( x_{95} \cdot \sqrt{DS} \cdot b = 3842492.54 \).

**Modeling the size of the insurance portfolio in the individual risk model**

The insurance company offers life insurance contracts for one year. Information regarding the coating structure is given in Table 7.

The relative protective allowance is \( \Theta = 20\% \). The number of contracts necessary to ensure the probability of ruin of the order of \( R = 5\% \) is determined.

\( N \) - the total number of contracts sold, \( X_i \) - the payments under the \( i \)-th contract, \( S = X_1 + \ldots + X_N \) - the total payments for the entire portfolio, \( \Theta \) - the relative protection premium. Then the premium for one contract is equal to:

\[ p = \left( 1 + \frac{\Theta}{100} \right) \cdot MX_i. \]

By setting

\[ P(S < N \cdot p) = 1 - \frac{R}{100}, \]

On the other side,

\[ P(S < N \cdot p) = \Phi \left( N \cdot p - MS \right) = \Phi \left( \frac{N \cdot p - 90}{\sqrt{DS}} \right) = \Phi \left( \sqrt{N} \cdot \frac{\Theta}{100} \cdot \frac{MX_i}{\sqrt{DX_i}} \right). \]

So

\[ \sqrt{N} \cdot \frac{\Theta}{100} = x_{95}. \]

Hence, for the desired number of contracts, we obtain:

\[ N \approx \frac{x_{95} \cdot DX_i}{\left( \frac{\Theta}{100} \right)^2 \cdot (MX_i)^2}. \]

Let’s find the values for \( MX \): the \( DX \)-individual contract:

\[ MX_i = 0.1 \cdot 500000 + 0.01 \cdot 1000000 = 60000, \]

\[ DX_i = (0.1 \cdot 500000^2 + 0.01 \cdot 1000000^2) - (60000)^2 = 314 \cdot 10^8. \]

Then

\[ N \approx \frac{x_{95}^2 \cdot DX_i}{\left( \frac{\Theta}{100} \right)^2 \cdot (MX_i)^2} = \frac{1.645^2 \cdot 314 \cdot 10^8}{(0.2)^2 \cdot (60000)^2} \approx 591. \]

### Table 7: Structure of insurance coverage.

<table>
<thead>
<tr>
<th>Sum insured</th>
<th>Cause of death</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a = 500000)</td>
<td>natural</td>
<td>(p = 0.1)</td>
</tr>
<tr>
<td>(b = 1000000)</td>
<td>accident</td>
<td>(q = 0.01)</td>
</tr>
</tbody>
</table>

**Reinsurance of risks and analysis of income of an insurance company**

The policyholder buys a group insurance contract for a group of \( N = 4 \) people. The insurer assigns a protective premium \( \Theta = 20\% \) and enters into a reinsurance contract for excessive individual losses with a deductible limit \( r = 1 \) for each risk. The relative protective premium used by the reinsurer is \( \Theta^* = 20\% \).

At the end of the term of the contract, the insurer calculates the balance of income and expenses. Revenues include premium and expenses consist of insurance claims paid (excluding reinsurer’s share), reinsurance fees and administrative costs of \( s = 5\% \) of premium (Volkov, 1982).

The value of the expected income of the insurer at the end of the contract period is determined if the distribution of individual losses is given in Table 8.

Let \( X_i \) - the amount of payments to the \( i \)-th insured, \( X_i' = \min(X_i, r) \) the share of the insurer, \( X_i'' = \max(X_i - r, 0) \) the share of the reinsurer in the insurance indignation to the \( i \)-th insured.

The distribution of random variables \( X_i' \) is \( X_i'' \):

\[ P(X_i' = 0) = 0.5, \quad P(X_i' = 1) = 0.35 + 0.15 = 0.5, \]
$P(X'' = 0) = 0.5 + 0.35 = 0.85, \ P(X' = 7) = 0.15$.

The expected loss of the reinsurer for one insured person is equal to:

$$MX'' = 0 \cdot 0.85 + 7 \cdot 0.15 = 1.05.$$  

Accordingly, the total expected losses of the reinsurer are equal to:

$$N \cdot MX'' = 4 \cdot 1.05 = 4.2.$$  

So, there is a fee for reinsurance protection

$$(1 + \theta^*) \cdot N \cdot MX'' = (1 + 0.2) \cdot 4.2 = 5.04.$$  

Let be $S' = X'_1 + X'_2 + X'_3 + X'_4$, the share of the insurer in the total losses. Let's find the distribution of this random variable. For this, its generating function is calculated:

$$Mz^{S'} = (Mz^X)^y = \left(0.5 + 0.5z\right)^4 = \frac{1}{16} \cdot \left(1 + z\right)^4 = \frac{1}{16} \cdot \left(1 + 4z + 6z^2 + 4z^3 + z^4\right) .$$  

The coefficients at powers of $z$ give the required distribution. It is shown in Table 9.

Since the total premium under insurance contracts is equal to $(1 + \theta) \cdot N \cdot MX = (1 + 0.2) \cdot 4 \cdot (0.5 + 0.5 + 1.05 + 8 - 0.15) = 7.44$, administrative costs are equal to $(1 + \theta) \cdot N \cdot MX', s = 5.04 \cdot 0.05 = 0.25$, the amount of income at the end of the contract is:

$$D = (1 + \theta) \cdot N \cdot MX' - (1 + \theta^*) \cdot N \cdot MX'' - (1 + \theta) \cdot N \cdot MX' \cdot s - S' =$$

$$= 7.44 - 5.04 - 0.372 - 2.028 - S' .$$  

The distribution of the random variable $D$ is obtained from the distribution of the random variable $S'$. It is shown in Table 10.

Then the average expected return of the insurer will be equal to

$$MD = 2.028 \cdot 0.0625 + 1.028 \cdot 0.25 + 0.028 \cdot 0.375 - 0.972 \cdot 0.25 - 1.972 \cdot 0.0625 = 0.028 .$$  

The ruin probability is:

$$P(D \leq 0) = 0.25 + 0.0625 = 0.3125 \sim 31.25% .$$

**Modeling the own retention limit for reinsurance risks**

The company concludes $N = 10,000$ life insurance contracts of the same type for a period of 1 year. The structure of insurance coverage is shown in Table 11.

The company sets the insurance fee based on the probability of ruin $R = 5\%$.

The insurance company intends to conclude an excessive loss reinsurance contract with a retention limit $r$ ( $a \leq r \leq b$).

The reinsurer company sets the relative premium equal to $\theta^* = 60\%$.

The value of the own retention limit $r$ is determined, which would minimize the probability that additional funds will need to be raised for payments on the portfolio under consideration (probability of ruin).

For calculations, it is convenient to use 100,000 AZN. as a
unit of monetary amounts, so that the payment $X_i$ under one contract takes the values $10$, $1$ and $0$ with probabilities $0.0005$, $0.002$ and $0.9975$, respectively. The average value of the payment under one contract is

$$MX_i = 0.0005 \cdot 10 + 0.002 \cdot 1 = 0.007,$$

while the variance

$$DX = MX_i - (MX_i)^2 = 0.0005 \cdot 100 + 0.002 \cdot 1 - 0.000049 = 0.051951.$$ 

Since the company sets the gross premium $p$ such that the probability of ruin is $5\%$, we have:

$$\theta = \frac{x_{0.05}}{\sqrt{N}} \cdot \frac{\sqrt{DX}}{MX_i} = \frac{1.645 \cdot 0.051951}{\sqrt{10000} \cdot 0.007} \approx 0.5359 \sim 53.59\%.$$ 

Thus, the net premium for one contract is $0.007$ conventional units, and the protective allowance $\theta = 53.59\%$, we have:

$$p = MX_i \cdot (1 + \theta) = 0.007 \cdot (1 + 0.5359) = 0.0107513$$

then the company’s fund will be $u = N \cdot p = 10000 \cdot 0.0107513 = 107513$ conventional units.

The company decides to reinsure claims exceeding $r$ manats, $100000 \leq r \leq 1000000$, with a reinsurance company. Since 100,000 AZN is used as a unit for measuring monetary amounts, $r$ varies from $1$ to $10$. In this case, the payment to the transmission company under one contract, $X'_i$, takes three values: $1$, $r$ and $0$ with probabilities $0.002$, $0.0005$ and $0.9975$, respectively. Its mean and variance are equal:

$$MX'_i = 1 \cdot 0.002 + r \cdot 0.0005 = 0.002 + 0.0005r,$$

$$DX'_i = 1 \cdot 0.002 + r^2 \cdot 0.0005 - (0.002 + 0.0005r)^2 = 0.002 + r^2 \cdot 0.0005.$$ 

The average value and variance of total payments for the entire portfolio, $S'$, is:

$$MS' = N \cdot MX'_i = 10000 \cdot (0.002 + 0.0005r) = 20 + 5r,$$

$$DS' = N \cdot DX'_i = 10000 \cdot (0.002 + 0.0005r) = 20 + 5r.$$ 

For a reinsurance company, the average value of the payment under one contract is:

$$MX''_i = 0.007 - 0.002 - 0.0005r = 0.005 - 0.0005r.$$ 

Therefore, the reinsurance fee for one contract is equal to:

$$MX''_i \cdot (1 + \theta) = (0.005 - 0.0005r) \cdot (1 + 0.6) = 0.008 - 0.0008r.$$ 

The total reinsurance fee for the entire portfolio is:

$$N \cdot (0.008 - 0.0008r) = 10000 \cdot (0.008 - 0.0008r) = 80 - 8r$$

and therefore, after reinsurance, the premium collected by the company will decrease to:

$$u' = u - (80 - 8r) = 107513 - 80 + 8r = 27513 + 8r.$$ 

For the probability $R'$ that the total payments of the insurance company, $S'$, is greater than the company's assets, $u'$, using the Gaussian approximation, we have:

$$\Theta = \frac{x_{0.95}}{\sqrt{N}} \cdot \frac{\sqrt{DS'}}{MX''_i} = \frac{1.645 \cdot \sqrt{0.051951}}{\sqrt{100000} \cdot 0.007} \approx 0.59,53 \sim 59.53\%.$$ 

Thus, if we want to minimize the probability $R'$, we need to choose the parameter $r$ in such a way that the function:

$$h(r) = \frac{(7.513 + 3r)^2}{20 + 5r^2}$$

took on the highest value. Insofar as:

$$h'(r) = 2 \left( \frac{7.513 + 3r}{2} \right) \frac{(12 - 7.513r)}{4 + r^2}$$

the optimal value is $r=1.605$, which in absolute terms corresponds to 160500 AZN.

Since $\sqrt{h(1.605)} \approx 2.15$, the probability of ruin at this holding limit is approximately $1.6\%$. The company's expected income before reinsurance was AZN $3,750,000$. (It is calculated as the difference between the collected premiums and the expected payments). After reinsurance, the company’s expected income was AZN 1,230,000. Thus, the decrease in the probability of ruin was achieved at the cost of reducing the expected income by AZN 2,520,000. We also note that in order to achieve the same ruin probability without reinsurance, it is necessary to increase the premium to the value:

$$MX''_i = 0.007 - 0.002 - 0.0005r = 0.005 - 0.0005r.$$ 

$$\Theta = \frac{x_{0.95}}{\sqrt{N}} \cdot \frac{\sqrt{DX''_i}}{MX''_i} = \frac{1.645 \cdot \sqrt{0.051951}}{0.007} \approx 0.01202.$$
conventional units, or AZN 1202, that is, by 12%. This means an increase in the total premium for the entire portfolio to AZN 12,020,000, and the company’s expected income to AZN 12,020,000. Of course, from the point of view of the insurance company, this is much better than AZN 1,230,000. expected income in case of purchasing reinsurance coverage, but we must not forget about market factors - policyholders may not agree to purchase a more expensive product.

Conclusion

In this study, within the framework of the individual risk model, mathematical modeling of the risk reinsurance process was carried out. Mathematical modeling was carried out:

- premium values in the individual risk model;
- the size of the insurance portfolio in the individual risk model;
- income of the insurance company in case of reinsurance of risks;
- own retention limit for reinsurance risks.

Based on the obtained mathematical models, models have been developed that allow to find the cost of an insurance policy and the size of the portfolio, analyze the income of an insurance company and determine the optimal limit of own retention for reinsurance risks.

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