Technology fluctuation and neighborhood efficiency of capital accumulation

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ABSTRACT

In this paper, geometric Brownian motion is employed with the purpose of introducing technology fluctuation into the underlying economy. In this case, it is confirmed that modified Golden Rule should be defined and derived based on the martingale path of capital-labor ratio. Finally, we supply a complete neighborhood characterization of the modified Golden Rule from both time and space dimensions. That is to say, we focus on neighborhood efficiency of capital accumulation in neoclassical stochastic growth models and our assertion actually implies that it is exactly the underlying technology fluctuation that induces neighborhood efficiency of capital accumulation, thereby demonstrating an intrinsic connection between the neoclassical capital accumulation and the technological fluctuation.

Key words: Martingale path, modified golden rule, Stochastic TFP, neighborhood efficiency.

JEL classification: C60; E13; E22.

INTRODUCTION

It is well-known that the Golden Rule path has been playing a very important role in neoclassical theory of capital accumulation (Cass, 1965, 1966, 1972; Samuelson, 1965; de la Croix and Ponthiere, 2010; Mitra and Ray, 2012; Acemoglu, 2012; Dai and Shen, 2013) starting from the pioneering papers of Phelps (1961, 1965). Usually, in deterministic cases, the Golden Rule is derived through the steady state or balanced path of capital accumulation (Cass, 1966, 1972). On the contrary, in permanently non-stationary environment, we argue that the modified Golden Rule can be appropriately defined based upon the martingale path of capital-labor ratio. Rather, it is a diffusion process with strong Markov property. Loosely speaking, one might, in certain sense, regard the Martingale path as the generalized balanced path of stochastic versions. Smith (1986) also has investigated the generalization of deterministic Golden Rule results in the case with resource uncertainty instead of the current technology fluctuation. Nevertheless, the so-called stochastic Golden Rule is derived by maximizing the expected utility of consumption at the stochastic steady state, which is of course different from the classical definition of Golden Rule, which actually corresponds to the maximization of consumption itself. Moreover, a closed-form solution is not derived there. Furthermore, Schenk-Hoppé (2002) is also motivated to study the Golden Rule in stochastic Solow growth models. However, Schenk-Hoppé employs dynamical systems theory, especially the concept of a random fixed point (Schenk-Hoppé and Schmalfuss, 2001), to prove the existence of a Golden-Rule savings rate for the stochastic Solow model. As can see later on, Schenk-Hoppé’s methodology cannot be applied to the present case and hence one can interpret our result as a nontrivial complementary-material to Schenk-Hoppé’s work.

In addition, we are in line with Wälde (2011), Bucci et al. (2011) and Dai (2014) which stated that geometric Brownian motion is employed to introduce technology fluctuation into the underlying economy. And we have shown that it is exactly the underlying technology...
fluctuation that induces neighborhood efficiency of capital accumulation, which can be regarded as the major contribution of the present investigation relative to existing studies focused on stochastic TFP. Definitely, we admit that existing studies also employ some other relatively different ways to introduce uncertain technology into growth models, for example, one may refer to Mirman (1972), Joshi (1997) and among others. So, the advantage of the present specification is that on the one hand, we can use this model to appropriately characterize some continuously small technology variances by a diffusion process while on the other hand, this specification makes things easier in establishing the closed-form solution of the modified Golden Rule capital-labor ratio.

Additionally, the present technology fluctuation can also be reasonably interpreted as a kind of realization of the so-called creative destruction first proposed by the greatest economist Joseph Schumpeter. In consequence, our result reveals that there exists a certain connection between the creative destruction and neoclassical capital accumulation, that is, technology fluctuation will induce rather than imposing restrictions on the efficiency of capital accumulation. Accordingly, our assertion actually has strengthened existing ideas from the perspective of capital accumulation. For example, Acemoglu and Robinson (2012) emphasize that it is frequent creative destruction and technological change that support sustainable economic growth. Baumol et al. (2007) vividly indicate that the Achilles’ heel of big-firm capitalism itself is the tendency not to innovate. And hence they claim that entrepreneurial capitalism is the system that is most conducive to radical innovation. Naturally, the corresponding economic implication can be expressed as follows, that is, we should explore and follow those institutional arrangements that will support or motivate such kind of technology fluctuation or technological change studied by the paper from the particular viewpoint of efficient capital accumulation and sustainable economic growth.

Indeed, it is proved that the path of capital-labor ratio will converge in the long run by employing the well-known Doob’s Martingale Convergence Theorem. Nevertheless, it is not ensured that the modified Golden Rule will be the global attractor or asymptotic turnpike of the martingale dynamics of capital accumulation, which in certain sense resembles to the conclusion, noted recently by Ray (2009) and Mitra and Ray (2012) that the well-known Phelps-Koopmans Theorem does not always hold true. Rather, the global attractor or asymptotic turnpike may be either above, equal or below the modified Golden Rule, which depends on the limit of the martingale path of capital-labor ratio.

Noting that the modified Golden Rule need not be the global attractor or asymptotic turnpike of the path of capital-labor ratio, one major goal of the paper is to supply a relatively complete characterization of the neighborhood properties of the modified Golden Rule. Luckily, it is shown that the modified Golden Rule will exhibit the so-called neighborhood turnpike property (Samuelson, 1965) from perspectives of both time and space dimensions, that is, the path of capital-labor ratio will ultimately enter the given neighborhood of the modified Golden Rule in a uniformly bounded time length, and meanwhile the invariant Borel probability measure (For the formal definitions of invariant Borel probability measure, and the conditions ensuring the existence and uniqueness of invariant probability measure, one may refer to Bayer and Wälde (2011) and some references therein for more details) deduced from the continuous time Markov process of capital accumulation will place nearly all mass close to the modified Golden Rule. One might say that the path will spend almost all the time and with a nearly close-to-one probability staying in some given neighborhood of the modified Golden Rule even though the path will not eventually converge to the modified Golden Rule. Consequently, we regard such kind of property as neighborhood efficiency of the stochastic dynamics of capital accumulation.

We certainly admit that the standard of neighborhood efficiency is not suitable for the evaluation of long-run efficiency of capital accumulation (Cass, 1972), it however provides us with a good benchmark for the short-term or midterm efficiency evaluation of capital accumulation. For example, such kind of standard would be suitable for those developing economies such as China (Song et al., 2011). What is more, the result shows that “capital over-accumulation” (Cass, 1972) and “excessive capital deepening” (Phelps, 1965) do not always imply dynamic inefficiency in a permanently non-stationary environment of neoclassical economies. In this sense, our result again confirms the widely-applicable argument that we cannot say whether or not a theory is correct in any absolute sense, only that it is better than others (Al-Najjar and Weinstein, 2008), especially in social sciences.

**ENVIRONMENT**

We establish our theory in a one-sector neoclassical model with stochastic growth. In particular, and without loss of any generality, it is assumed that the uncertain technological progress is described as solely labor-augmenting that is, we employ the following neoclassical production function:

\[ Y(t) = F\left(K(t), A(t)L(t)\right). \]

(1)

which is assumed to be strictly concave, homogenous of first degree and exhibit constant returns to scale effect with \( K(t) \) denoting the aggregate capital stock and \( L(t) \) representing the labor force or population size. And the law of motion of capital accumulation is expressed as follows:
growth since the reform started in 1978.

MODIFIED GOLDEN RULE AND NEIGHBORHOOD EFFICIENCY

We put the drift term in Equation (4) that:

\[ s(k(t)) f(k(t)) - \left( \delta + n + \beta - \sigma^2 \right) k(t) = 0. \]

(5)

which yields the consumption per capita:

\[ c(t) = f (k(t)) - \left( \delta + n + \beta - \sigma^2 \right) k(t). \]

(6)

Hence, by Equation (6), we obtain the following first order condition:

\[ \frac{\partial c(t)}{\partial k(t)} = f'(k(t)) - \left( \delta + n + \beta - \sigma^2 \right) = 0 \]

by which we define.

**Definition 1 (Modified Golden Rule):** By Equation (5), we get the martingale path of capital-labor ratio from Equation (4), and the corresponding modified Golden Rule \( k^* \) is determined by \( f'(k^*) = \delta + n + \beta - \sigma^2 \) or \( k^* = (f')^{-1}(\delta + n + \beta - \sigma^2) \).

**Remark 1:** It is especially worthwhile mentioning that the modified Golden Rule given in Definition 1 is completely compatible with the Golden Rule in deterministic economies. Moreover, it is interesting to note from (5) that the modified Golden Rule relies on the dynamics without linear growth as in deterministic cases while definitely with constant fluctuation. In other words, only the long-run trend term of capital accumulation is excluded and the underlying economy still lies in a constantly stochastic environment. As a consequence, the modified Golden Rule sufficiently reflects the information of technology fluctuation in the economy, and it successfully extends the classical concept of Golden Rule widely used in deterministic economies.

Substituting Equation (5) or Equation (6) into Equation (4) shows that:

\[ dk(t) = -\sigma k(t) dB(t). \]

(7)

If one finds in other stochastic models that the path of capital-labor ratio is a martingale, then the martingale path
will have a similar form like Equation (7) by using Martingale Representation Theorem. Now, we derive:

**Proposition 1:** Based upon the martingale path of capital-labor ratio given by Equation (7), we get that $\dot{k}(t)$ is uniformly bounded for $t \geq 0$ for almost all $\omega \in \Omega$. Moreover, it is confirmed that:

$$\sup_{t \geq 0} \mathbb{E}\left[ (\lambda + k_0)k_0 < \infty \right].$$

(8)

and,

$$\mathbb{E}\left[ (\lambda + k_0^*)^2 \right] \leq (k_0 + k_0^*)^2 + (k_0 - k_0^*)^2 < \infty.$$ 

(9)

for $\forall t \geq 0$ and $\forall 0 < \lambda < \infty$

**Proof:** See Appendix A. ■

Hence, it is easily seen that:

**Corollary 1:** The path of capital-labor ratio $k(t)$ converges in $L^1(\mathbb{P})$, and the corresponding limit, denoted by $\hat{k}$, belongs to the space $L^1(\mathbb{P})$.

**Proof:** Combining Proposition 1 with the Doob’s Martingale Convergence Theorem gives the required assertion. ■

**Remark 2:** The limit could be regarded as the “asymptotic turnpike” or “stochastic global attractor” of the path of capital accumulation given by Equation (7). Moreover, one may check the well-known Phelps-Koopmans Theorem (Ray, 2009; Mitra and Ray, 2012) in the underlying economy by comparing the limit with the modified Golden Rule. However, the value of the limit usually cannot be explicitly derived in constantly non-stationary environments. Moreover, one can see that:

**Corollary 2:** Provided the modified Golden Rule $k^*$ given by Definition 1, we see that:

$$\lim_{t \to \infty} \mathbb{E}\left[ k(t) - k^* \right] \leq \mathbb{E}\left[ \dot{k} - k^* \right].$$

**Proof:** This is just an application of Corollary 1, Triangle Inequality and Lebesgue Dominated Convergence Theorem.

**Remark 3:** Corollary 2 implies that the expected or average distance between the stochastic path of capital accumulation and the modified Golden Rule is strictly bounded by a deterministic value.

Now, we are in the position to show the neighborhood characterization of the modified Golden Rule and we first give the following assumption:

**Assumption 1:** It is assumed that $k(t) \in \mathbb{R}_+ = [0, \infty]$, which is the one point compactification of $\mathbb{R}$ at infinity with the induced topology, for $\forall t \geq 0$. Moreover, it is supposed that there exists a unique invariant Borel probability measure $\mu$ on $\mathbb{R}_+$ such that $\mu([0] \cup \{+\infty\}) = 0$.

**Remark 4:** Mirman (1972) constructs a one-sector growth model with a discrete-time Markov process of capital stock. And Theorem 1 of Mirman confirms that there exists a stationary probability measure that has no mass at either zero or infinity. In contrast, the present paper constructs continuous time Markov process of capital-labor ratio. Nonetheless, one can still prove that there exists a unique invariant Borel probability measure satisfies the requirements of Assumption 1 under certain relatively weak conditions. For more details, one may refer to Theorem 2 of Imhof (2005), Theorem 3 of Benaim et al. (2008) and Theorem 5 of Schreiber et al. (2011). The present paper omits the corresponding proof just for the sake of simplicity. About this issue in economics, there are some classical literatures such as Brock and Mirman (1972), Futia (1982) and Hopenhayn and Prescott (1992). And about recent development in this area, one can refer to Nishimura and Stachurski (2005).

Immediately, it follows from Assumption 1 that:

**Proposition 2:** (Ergodic Theorem): If $k(0) \equiv k_0$ is initially distributed like an invariant Borel probability measure $\mu$, then we get that:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T k(t) dt = \int_{\mathbb{R}_+} k \mu(k) dk.$$ 

almost surely $\mathbb{P}$.

**Proof:** Application of the well-known Birkhoff Ergodic Theorem gives the required result. ■

**Remark 5:** Proposition 2 shows that the time average of the capital-labor ratio will converge to the space average of the capital-labor ratio almost surely, which yields the ergodic property of the martingale path of capital-labor ratio given by Equation (7). Moreover, one can also interpret...
Proposition 2 as a Law of Large Numbers of stochastic capital accumulation in some sense.

Now, we can establish,

**Theorem 1** (Neighborhood Efficiency): There exists a constant \( \xi > 0 \) such that for \( \forall \alpha > 0 \) with \( \alpha^2 > \xi \),

\[
E\left[ \tau_{\beta_{\alpha}(k^*)}(\omega) \right] \leq \frac{\text{dist}\left(k_0, k^*\right)}{\alpha^2 - \xi};
\]

(i) \( \mu[\bar{B}_{\alpha}(k^*)] \geq 1 - \frac{\xi}{\alpha^2} + 1 - \varepsilon \)

(ii) \( \text{dist}\left(k_0, k^*\right) = k^* \log\left(k^*/k_0\right) \) for \( k_0 = k(0) > 0 \)

where \( B_{\alpha}(k^*) = \{ k(t) \in \mathbb{R}; |k(t) - k^*| < \alpha, t \geq 0 \} \),

\[ \tau_{\beta_{\alpha}(k^*)}(\omega) = \inf \left\{ t \geq 0; k(t) \in \bar{B}_{\alpha}(k^*) \right\} \] and

\[ \text{dist}\left(k_0, k^*\right) = k^* \log\left(k^*/k_0\right) \text{ for } k_0 = k(O) > 0 \]

**Proof:** See Appendix B. ■

**Remark 6:** This proof brings the method employed by Imhof (2005) and Dai (2012). And the proof itself is of independent interest in proving neighborhood types of turnpike theorems. Theorem 1 gives a neighborhood characterization of the modified Golden Rule from both time (Markov time) and space (invariant Borel probability measure) dimensions. As a result, we argue that the martingale path of capital-labor ratio defined in Equation (7) exhibits neighborhood efficiency: the path of capital-labor ratio will ultimately enter the given neighborhood of the modified Golden Rule in a uniformly bounded time length, and meanwhile the invariant Borel probability measure deduced from the continuous time Markov process of capital accumulation will place nearly all mass close to the modified Golden Rule. Moreover, we argue that technology fluctuation induces neighborhood efficiency in the underlying economy.

What's the potential application of our theoretical result? It seems really hard to see any direct application of our abstract assertion, we, however, will offer the following implicit implications to reveal the potential applied-value of our formal argument. First, notice that too much work emphasizes the long-run efficiency of capital accumulation, which is definitely important, we thus provide a new standard from the standpoint of short-term accumulation, which certainly is quite necessary for supplying acceptable and sufficient economic incentives for those newly emerging economies. There are two major reasons supporting this argument. On the one hand, newly emerging economies usually imitate existing technologies from developed economies, and the corresponding technological fluctuation is generally continuously small, which thus coincides with our assumption. On the other hand, in reality, efficiency evaluation itself implies certain computation and operation cost and newly emerging economies have adequate incentives to employ the most direct and effective way to complete such a work. So, our result established in Theorem 1 has its congenital advantage because of its objective characteristic, relatively high accuracy and relying on very low dimension of exogenous parameters.

Secondly, as can be seen from Theorem 1, our result also implies a speed standard characterized by a stopping time, that is, the minimum time required to enter the given neighborhood of modified Golden Rule from any given initial state. This is definitely a vivid characteristic of our result compared to existing studies. Interestingly, this finding directly leads us to a much more comprehensive philosophy when we are motivated to comparatively analyze capital accumulation of different economic systems. For example, there are two economies with different levels of modified Golden Rule, e.g., the first one is relatively bigger than the second one. Hence, we usually claim that the first one will definitely and economically dominate the second one. Nonetheless, our result will argue that this conclusion is really hasty and hence not comprehensive, and it even does not make any sense. Why? When we attempt to evaluate the potential of capital accumulation for different economies, we should also consider the efficiency from the time aspect, that is, the first economy may take 15 years to reach its neighborhood efficiency, whereas the second one only takes 5 years. As a result, Theorem 1 confirms that both the height of our goal and the speed leading to finally achieve our goal are equivalently crucial from the perspective of economic efficiency.

Not only that, we are encouraged to add the following additional comment for Theorem 1. We hope to clearly indicate that there exists an intriguing relation between our major result and the concept of flexibility, which has attracted economists' interest. Attention to understand it in the category of economics (Amador et al., 2006; Huang, 2008). In fact, we understand the concept of flexibility under the current background like this, that is, it seemingly ingeniously captures the dynamic tradeoff between evaluation accuracy and sustainable economic incentive. To be exact, the selected scope or radius of the given neighborhood completely determined by the exogenous parameter \( \alpha \) reflects the underlying flexibility of the present evaluation mechanism proposed by Theorem 1. In particular, if we are to pursue a relatively high goal of modified Golden Rule, then we can properly extend the given neighborhood; symmetrically, if the goal is relatively small, then we can proportionally narrow the neighborhood. Therefore, we are kept in a subtle balance between the evaluation accuracy and the sustainable economic incentive. As is broadly recognized, accuracy is important because it reveals useful information of the real macroeconomic process and meanwhile avoids any unnecessary overconfidence, while the intrinsic economic incentive is sustainable only when there are necessary
external encouragements from objective accomplishments. In sum, policy makers should absolutely carefully sustain such a balance between the both. It, therefore, can be regarded as an insightful lesson policy makers may have learned from our model.

Concluding remarks

In this study, we investigate neoclassical theory of capital accumulation in a permanently non-stationary environment. To the best of our knowledge, we have, for the first time, derived the modified Golden Rule based upon the martingale path of capital-labor ratio that has captured the information of technology fluctuation of the underlying economy. Moreover, neighborhood characterization of the modified Golden Rule has been supplied from both time, that is, Markov time, and space, that is, invariant Borel probability measure, dimensions. And it is especially worth emphasizing that neighborhood efficiency of stochastic capital accumulation does not depend on the specification of production function. It is further believed that such kind of efficiency standard will be suitable for those developing economies mainly focusing on short-term economic growth. Finally, it would be of independent interest to employ geometric Lévy process instead of the geometric Brownian motion to capture technology breakthrough (by positive jump term) in reality. Since the underlying technique regarding geometric Lévy process has been thoroughly explored and analogously developed by Dai and Shen (2012), Dai (2013a, b) and Dai et al. (2013), the required technique can be easily found in these papers and hence the corresponding verification as well as comparative analysis has been left to the interested reader.

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APPENDIX

A. Proof of Proposition 1

It follows from Equation (7) that \( k(t) \) will be a \( F_t \)-martingale w. r. t. \( P \). Thus, by the Doob’s Martingale Inequality, we obtain:

\[
P\left( \sup_{0 \leq t \leq T} |k(t)| \geq \lambda \right) \leq \frac{1}{\lambda} \mathbb{E} \left[ |k(T)| \right] = \frac{k_0}{\lambda}, \quad \forall \lambda > 0 , \quad \forall T > 0.
\]

Without loss of generality, we put \( \lambda = 2^m \) for \( m \in \mathbb{N} \), then:

\[
P\left( \sup_{0 \leq t \leq T} |k(t)| \geq 2^m \right) \leq \frac{1}{2^m} k_0, \quad \forall m \in \mathbb{N}, \forall T > 0.
\]

Using the well-known Borel-Cantelli Lemma, we arrive at:

\[
P\left( \sup_{0 \leq t \leq T} |k(t)| \geq 2^m \right) \to 0, \quad \forall m \in \mathbb{N}, \forall T > 0.
\]

So for a.a. \( \omega \in \Omega \), there exists \( \bar{m}(\omega) \in \mathbb{N} \) such that:

\[
\sup_{0 \leq t \leq T} |k(t)| < 2^m \quad \text{a.s. for } m \geq \bar{m}(\omega), \quad \forall T > 0.
\]

That is:

\[
\lim_{T \to \infty} \sup_{0 \leq t \leq T} |k(t)| \leq 2^m, \quad \text{a.s. for } m \geq \bar{m}(\omega).
\]

Thus, \( k(t) = k(t, \omega) \) is uniformly bounded for \( t \in [0, T] \), \( \forall T > 0 \) and for a.a. \( \omega \in \Omega \). Moreover, by Kolmogorov’s Inequality, we get:

\[
P\left( \sup_{0 \leq t \leq T} |k(t)| \geq \lambda \right) \leq \frac{1}{\lambda^2} \mathbb{E} \left[ \left[ k(T) \right]^2 \right], \quad \forall 0 < \lambda < \infty, \quad \forall T > 0.
\]

It follows form (A.1) that:

\[
\frac{1}{\lambda^2} \mathbb{E} \left[ \left[ k(T) \right]^2 \right] \leq \frac{k_0}{\lambda} \iff \mathbb{E} \left[ \left[ k(T) \right]^2 \right] \leq \lambda k_0, \quad \forall T > 0.
\]

(A.2)

Noting that:

\[
\mathbb{E} \left[ \left[ k(T) \right]^2 \right] = \mathbb{E} \left[ \left[ k(T) \right]^2 \right] - (k_0)^2, \quad \forall T > 0.
\]

We get by (A.2):

\[
\mathbb{E} \left[ \left[ k(T) \right]^2 \right] \leq (\lambda + k_0)k_0 < \infty, \quad \forall 0 < \lambda < \infty, \quad \forall T > 0.
\]

which yields:

\[
\sup_{T > 0} \mathbb{E} \left[ \left[ k(T) \right]^2 \right] \leq (\lambda + k_0)k_0 < \infty.
\]
as required in Equation (8). Noting that \( k(t) - k^* \) will be a \( F_t \)-martingale w. r. t. \( P \), applying the Doob's Martingale Inequality and Triangle Inequality implies that:

\[
P \left( \sup_{0 \leq s \leq t} |k(s) - k^*| \geq \lambda \right) \leq \frac{E \left[ \left| k(T) - k^* \right|^2 \right]}{\lambda} \leq \frac{k_0 + k^*}{\lambda},
\]

\( \forall \lambda > 0, \forall T > 0. \) \hspace{1cm} (A.3)

On the other hand, by Kolmogorov's Inequality, we get:

\[
P \left( \sup_{0 \leq s \leq t} |k(s) - k^*| \geq \lambda \right) \leq \frac{1}{\lambda^2} \text{var} \left[ k(T) - k^* \right],
\]

\( \forall 0 < \lambda < \infty, \forall T > 0. \) \hspace{1cm} (A.4)

Hence, combining (A.3) with (A.4) reveals that:

\[
\frac{1}{\lambda^2} \text{var} \left[ k(T) - k^* \right] \leq \frac{k_0 + k^*}{\lambda} \Leftrightarrow \text{var} \left[ k(T) - k^* \right] \leq \lambda (k_0 + k^*), \forall T > 0. \] \hspace{1cm} (A.5)

Noting that:

\[
\text{var} \left[ k(T) - k^* \right] = E \left[ k(T) - k^* \right]^2 - \left( E \left[ k(T) - k^* \right] \right)^2,
\]

\( \forall T > 0 \). \hspace{1cm} (A.6)

And by Minkowski-Riesz Inequality, we obtain:

\[
E \left[ k(T) - k^* \right] \geq E \left[ k(T) \right] - E \left[ k^* \right] = k_0 - k^*,
\]

i.e.,

\[
\left( E \left[ k(T) - k^* \right] \right)^2 \geq \left( k_0 - k^* \right)^2.
\]

Inserting this into (A.6) reveals that:

\[
\text{var} \left[ k(T) - k^* \right] \leq E \left[ \left( k(T) - k^* \right)^2 \right] - \left( k_0 - k^* \right)^2,
\]

\( \forall T > 0 \).

Combining this with (A.5) leads us to:

\[
E \left[ \left( k(T) - k^* \right)^2 \right] \leq \lambda (k_0 + k^*) + (k_0 - k^*)^2 < \infty,
\]

\( \forall 0 < \lambda < \infty, \forall T > 0 \).

which shows that \( k(t) - k^* \) is a square-integrable martingale and this gives the desired result in (9). Thus, the proof is complete. \( \blacksquare \)

B. Proof of Theorem 1

Given the SDE defined by (7), we can define the following characteristic operator of \( k(t) \):

\[
A_g (k_0) = \frac{1}{2} \sigma^2 (k_0) \frac{\partial^2 g}{\partial (k_0)^2} (k_0)
\]

for any \( k_0 = k(0) \geq 0 \). We now define Kullback-Leibler type distance (Bomze, 1991; Imhof, 2005) between \( k_0 \) and \( k^* \) as follows:

\[
g(k_0) = \log \left( \frac{k^*}{k_0} \right) \geq 0.
\]

Then we get

\[
A_g (k_0) = \frac{1}{2} \sigma^2 k^*.
\]

Noting from Equation (9) that there exists a constant \( \Sigma < \infty \) such that

\[
| k(t) - k^* |^2 \leq \Sigma, \forall t \geq 0,
\]

thus we have:

\[
A_g (k_0) \leq \frac{1}{2} \sigma^2 k^* + \Sigma - | k(t) - k^* |^2 \equiv \xi - | k(t) - k^* |^2.
\]

(B.1)

Define \( B_\alpha (k^*) = \{ k(t) \in \mathbb{R}_+, | k(t) - k^* | < \alpha, \forall t \geq 0 \} \) and,

\[
\tau^\prime (\omega) = \tau_{B_\alpha (k^*)} (\omega) = \inf \{ t \geq 0, k(t) \in \overline{B}_\alpha (k^*) = dB_\alpha (k^*) \}
\]

where \( \overline{B}_\alpha (k^*) \) denotes the closure of \( B_\alpha (k^*) \). Suppose that \( \alpha^2 > \xi \), for every \( k(t) \notin \overline{B}_\alpha (k^*) \), that is, \( k(t) \notin \overline{B}_\alpha (k^*) \), we get \( A_g (k_0) \leq - \alpha^2 + \xi \) by (B.1). Then by Dynkin’s formula:

\[
0 \leq E \left[ g \left( k(t \wedge \tau^\prime) \right) \right] = g(k_0) + E \left[ \int_0^{\tau^\prime} \frac{\partial g(k(s))}{\partial s} ds \right] \leq g(k_0) + (\xi - \alpha^2) E \left[ \tau^\prime (\omega) \right]
\]

Since \( t \wedge \tau^\prime \to \tau^\prime \) as \( t \to \infty \), then by Lebesgue Monotone Convergence Theorem we obtain

\[
0 \leq g(k_0) + (\xi - \alpha^2) E \left[ \tau^\prime (\omega) \right],
\]

which produces:
E[τ_{g,k}(ω)] = E[τ^*(ω)] ≤ \frac{g(k_0)}{\alpha^2 - \varepsilon} = \frac{\text{dist}(k_0, k^*)}{\alpha^2 - \varepsilon}.
\]

as required in (i). Moreover, for some constant \( k \), set up:

\[\tau_W = \tau_W(ω) = \inf\{t ≥ 0; g(k(t)) = W\} \]

Thus, by Dynkin's formula and inequality in (B.1):

\[0 ≤ E[g(k(t ∧ \tau_W))] = E[g(k_0)] + E\left[\int_{0}^{t∧\tau_W} A g(k(s)) ds\right] ≥ g(k_0) - E\left[\int_{0}^{t∧\tau_W} (k(s) - k^*)^2 ds\right] + E\left[t∧\tau_W\right] \xi \]

Notice that \( t∧\tau_W(ω) → t \) as \( W → \infty \), and by applying the well-known Lebesgue Bounded Convergence Theorem and Levi Lemma:

\[0 ≤ g(k_0) - E\left[\int_{0}^{\tau_W} (k(s) - k^*)^2 ds\right] + \xi t \]

which yields:

\[\lim_{t→∞} E\left[\frac{1}{t}\int_{0}^{t} (k(s) - k^*)^2 ds\right] ≤ \xi \]

(B.2)

If we let \( B_0^C(k^*) \) denote the indicator function of set \( B_0^C(k^*) \), then by (B.2) and Assumption 1, we arrive at:

\[\mu\left[B_0^C(k^*)\right] = \lim_{t→∞} E\left[\frac{1}{t}\int_{0}^{t} \chi_{B_0^C(k^*)}(k(s)) ds\right] \leq \frac{\xi}{\alpha^2}.\]

which implies

\[\frac{1}{\alpha^2} ≤ \mu\left[B_0^C(k^*)\right] \leq -\frac{1}{\alpha^2} \]

as required in (ii). ♦

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