In some geometric properties fixed point theory in non-expanding functions

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ABSTRACT

The aim of this study is to determine whether a sequentially weak continuous duality function can be included into a Banach space. It is met with weak limits by means of the norm and the space. It has a normal structure in the sense of Brodskii-Milman. This result of geometric reality allows for some associations. Fixed point theory for both single-valued and multi-valued functions indicates non-expanding matches.

Key words: Metric space, fixed point iterations, convergence speed, fixed point.
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INTRODUCTION

Let X be a nonempty weakly compact convex subset of a real Banach space X and let T be a nonexpansive mapping of X into it’s nonempty compact subsets:

\[ d(Tx,Ty) \leq d(x,y) = \|x-y\| \]

For all x,y \in X where d(x,y) denotes the Hausdorff metric. While the question of the existence of a fixed point for T remains open, several positive results were proved recently under various conditions of geometric type on the norm of X. The terms considered are listed below:

1.1 X is a sequentially continuous duality function:

\[ F_\varphi : X, \sigma(X,X^*) \rightarrow X^*, \sigma(X^*) \]

A function F_\varphi such that:

\[ <x,F_\varphi(x)> = \|x\| \|F_\varphi(x)\| \text{ and } \|F_\varphi(x)\| = \varphi(\|x\|) \]

for all x \in X, where \( \varphi : R^+ \rightarrow R^+ \) is continuous strictly increasing with \( \varphi(0) = 0 \) and \( \varphi(\infty) = \infty \). (Browder, 1972)

1.2. Every weakly compact convex subset Hof X has normal structure for each convex subset Lof H which contains more than one point there exists x \in L such that:

\[ \sup\{\|x - y\| : y \in L\} < \sup\{\|u - v\| : u, v \in L\} \]

(Brodskii and Milman, 1948)

1.3. If a sequence \{x_n\} converges weakly in X to x_0, then \lim\inf\|x_n - x\| > \lim\inf\|x_n - x_0\| \]

for all x \neq x_0. (Opial, 1967)

When T is single-valued, the existence of a fixed point for T in X was proved by Browder (1972) if X satisfies (1.1) and if T can be extended outside if it is nonexpansive way, and by Assad and Kirk (1972) if X satisfies (1.2).

A similar situation occurs in the multivalued case where one also encounters two different approaches; one by Browder (1972) who proved a fixed point theorem under condition (1.1) and some additional assumptions, and another by the Lami (1973) who obtained the same conclusion under condition (1.3).

It is a consequence of our main theorem that in both cases the third approximation is more general than the first approximation.

Theorem 1. 1. (1.1) implies (1.2) and (1.3) implies (1.2). No converse implication holds, not even when X and X^* are supposed to be uniformly convex.

Although (1.3) does not imply (1.1), there is some result in that direction, which supports the feeling that the gap between (1.3) and (1.2) is much deeper than that between (1.1) and (1.3).
To state this result I am define (1.1) as (1.1) except that \( F\varphi \) is only required to be sequentially continuous at zero and (1.3) as (1.2) except this \( \geq \) symbol is greater than or equal to the \( \geq \) symbol is replaced.

Theorem 1.2. (1.1) implies (1.3). The converse implication holds when the norm of \( X \) is uniformly Gateaux differentiable.

The first and second theorems are proved in chapter 2 and chapter 3, respectively. Some relevant results and a few examples that shed more light on the connections between these geometric properties are presented in Chapter 4. In chapter 5, we show that the space \( c_0 \) is equipped with Day’s norm, locally uniformly convex (Rainwater, 1969), does not satisfy (1.2). This example should be linked to the known facts that all proper convex spaces satisfy (1.2), as well as spaces where Day and Lovaglia have shown that they are locally uniformly convex but not isomorphic to any regular convex space (Yildirim and Özdemir, 2009; Moreau, 1967; Elmas and Hızarci, 2015).

**MAIN RESULTS**

To prove Theorem 1.1 we need two lemmas about the duality map

\( f_\varphi : X \to X^* \)

defined by

\[ <x, x^*> = \|x\| \|x^*\| \text{ and } \|x^*\| = \varphi(\|x\|) \]

for all \( x \in X \).

In this context condition (1) asserts the existence for some gauge \( \varphi \) of a sequentially weakly continuous selection for \( f_\varphi \).

The first lemma follows from the monotonicity of \( f_\varphi \) it has been extended in to general monotone operators (Gossez, 1970).

Theorem 2.1. If \( X \) satisfies 1.1 for some \( \varphi \), then \( f_\varphi \) is univalent for any \( \varphi \).

Proof.2.1. The monotone operator \( F\varphi \) is hemi-continuous by (1), thus maximal monotone by Minty’s classical argument. Since \( f_\varphi \) is also monotone we must have \( f_\varphi(x) = F\varphi(x) \).

That \( f^\mu \) is for another indicator, one valued \( \mu \) subtracts from the equation:

\[ f^\mu(x) \cdot f_\varphi(x) = \mu(\|x\|) = \varphi(\|x\|). \]

The Theorem 2.2 uses the observation of Asplund (1967) that \( f_\varphi(x) \) is the subdiﬀerential of the convex function \( \varphi(\|x\|) \) where:

\[ \Phi(s) = \int_0^s \Phi(t)dt \]

For \( \forall x^* \in X^* \text{ and } \forall y \in X, \Phi(\|y\|) \geq \Phi(\|x\|) + <y - x, x^*>. \]

Theorem 2.2. If \( f_\varphi(x) \) is univalent:

\[ \Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 <y, f_\varphi(x + sy) > ds \]

for all \( x, y \in X \).

Proof.2.2. If \( f_\varphi \) is single valued then \( f_\varphi \) is the Gateaux gradient of \( \Phi(\|x\|) \); this is due to a general result in the theory of convex functions (Moreau, 1967). However, it can be easily verified that \( f_\varphi \) is semi-continuous. Consequently, the theorem 2.2 simply refers to the fact that a function of a real variable is the integral of the continuous derivative.

Proof of Theorem 1.1 let’s suppose (1.1) and let \( x_n \to x_0 \).

Theorem 2.1 and theorem 2.2,

\[ \Phi(\|x_n - x\|) = \Phi(\|x_n - x_0\|) + \int_0^1 <x_0 - x, f_\varphi((x_n - x_0) + s(x_0 - x)) > ds \]

for all \( x \in X \).

\[ \lim \inf \Phi(\|x_n - x\|) = \lim \inf \Phi(\|x_n - x_0\|) + \int_0^1 \|x_0 - x\| \Phi(s(\|x\| - \|x\|)) \]

Sequential weak continuity of \( f_\varphi \) and dominant convergence give theorem

\[ \lim \inf \Phi(\|x_n - x\|) = \lim \inf \Phi(\|x_n - x_0\|) + \int_0^1 \|x_0 - x\| \Phi(s(\|x\| - \|x\|)) \]

An inequality that clearly demonstrates condition (1.3). The evidence (1.3) implies (1.2) is based on a characterization. The normal structure given in (Gossez and Lami, 1969) \( X \) satisfies (1.3) if and only if \( X \) does not contain a diametral sequence \( \{x_n\} \) weakly converging to zero (a non-constant array with).

\[ d(x_n, \text{ com} \{x_1, x_2, \ldots, x_{n-1}\}) \to \delta(\{x_n\}) \]

(2)

where \( d(x_n, \text{ com} \{x_1, x_2, \ldots, x_{n-1}\}) \) denotes the distance of \( x_n \) to the convex hull of \( \{x_1, x_2, \ldots, x_{n-1}\} \) and \( \delta(\{x_n\}) \) the diameter of \( \{x_n\} \).

Let’s imagine that (1.3) does not hold and take such a sequence. It follows from (2) that:

\[ \lim \|x_n - x\| = \delta(\{x_n\}) \]

for all \( y \in \text{ com} \{x_n\} \), for so everyone \( y \in \text{ com} \{x_n\} \). Taking \( y = 0 \), we get:

\[ \lim \|x_n\| = \delta(\{x_n\}) \]

This contradicts (1.3). Let us now return to the last part of
Theorem 1.1.

When $1 < p < \infty$, $p \neq 2$, $L^p(0, 1.2)$ satisfies (1.2) since it is uniformly convex, but Opial (1967) showed that even (1.3) does not hold.

**EXPLANATION**

3.1. A finite dimensional space whose norm is not differentiable provides another example of a space satisfying (1.1), (1.3) but not (1.1) by Theorem 2.1.

3.2. In the Hubert space case when $\varphi(s) = s$, estimation (1) reduces to an estimation obtained by Opial (1967).

3.3. A partial discussion: The following simple theorem will be needed, which proceeds the proof in sequence.

Theorem 3.1 Conditions (1.3) and (1.3'), are respectively equivalent to the analogous conditions obtained by replacing $\liminf$ by $\limsup$. 

Proof 3.1. Proof of Theorem 1.2. Assume (1.1) and let $x_n \to x_0$. As $f_0(x)$ is the subdifferential of $\Phi(\|x\|)$ we have:

$$\Phi(\|x_n - x\|) \geq \Phi(\|x_n - x_0\|) + \langle x_0 - x, f_0(x_n - x_0) \rangle$$

for all $x \in X$.

$$\liminf \Phi(\|x_n - x\|) \geq \limsup \Phi(\|x_n - x_0\|)$$

an inequality that clearly indicates the condition (1.3').

In the second part of Theorem 1.2, uniform assumption Gateaux differentiability is equivalent to $f_0$ condition be univalued and continuous on $X$, $\sigma(X, X^*)$, Norm ($\|\| \|$) into $X^*$, uniform in limited sets; In this view, we will show that if (1.3') is valid for any then $\Phi, f_0$ is sequentially weak continuous at zero. Let $x_n \to x_0$ and assume that $f_0(x_n)$ does not converge to zero for $\sigma(X, X^*)$.

Then there exist $\in X$ a subsequence $\{x_p\}$ such that:

$$< t, f_0(x_p)> \rightarrow < t, x^* > \neq 0$$

Describing

$$g(x) = \limsup \Phi(\|x_p - x\|)$$

g(x) is a continuous convex function over $X$ assuming its minimum at $x = 0$ by condition (1.3') and Theorem 3.1. From here:

$$y < 0 \forall x, y \in X \quad \frac{1}{|y|} g(yx) - g(0) \geq 0$$

and thus

$$\limsup \frac{1}{|y|} (\Phi(\|x_n - yx\|) - \Phi(\|x\|)) \geq 0$$

as $f_0$ is the subdifferential of $\Phi(\|x\|)$.

$$\limsup < x, f_0(x_n - yx) > = a \geq 0$$

Allowing the $|y| = 0$ bounds can be replaced by the smooth continuity of $f_0$ and consequently

$$\limsup < x, f_0(x_n) > = b \geq 0 \quad \forall x \in X.$$

Especially $< t, x^* > = 0$ there will be a contradiction.

We do not know whether the differentiability hypothesis in Theorem 1.2 is necessary.

**Different links**

Let’s start this paragraph by showing that the following current situation applies in general.

$$\begin{align*}
\text{(1.1)} & \quad \text{(1.2)} \\
\text{(1.1')} & \quad \text{(1.3)}
\end{align*}$$

We can show the above work as a function. Consider the space $L^2$ endowed with

The norm:

$$\|x\| = \max \left\{ \frac{1}{2} \|x\|L^2; \|x\|L^\infty \right\}$$

Different suggestions and studies can be presented to this study. Open to suggestions and criticism.

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