On estimation of the shape parameter of Lomax exponential distribution under uniform and Jeffery priors with medical applications

ABSTRACT

The main objective of the study is to develop a Bayesian estimator of a shape parameter of the Lomax Exponential distribution under a Uniform and Jeffery priors with four different loss functions. The four loss functions are Square error loss function (SEL), Quadratic error loss function (QEL), Weighted error loss function (WEL), and the Precautionary error loss function (PEL). By assuming a known value of a scale parameter, we derive the posterior distribution corresponding to each loss function. The Bayes estimators and their MSE are evaluated and compared using simulation and a real data set. Based on the numerical computation and graphical discussion, it is evident that the Quadratic loss function provides the best result as compared with others under a Uniform and Jeffery prior.

Key words: Lomax Exponential distribution, uniform prior, Jeffery prior, SELF, QELF, WELF, PELF, Bayes estimates, and MSE.

INTRODUCTION

The probability distributions are useful techniques to tackle the problem of decision making under uncertainty by choosing an appropriate value of the parameters involved in the probability function. The Bayesian estimator is the one that provides the best value for the unknown parameters in the model. The Medical field is richer with data sets skewed to the right and thus the choice of the positively skewed distributions under the Bayesian paradigm in making inferences would be more appropriate.

In a diagnostic test, it is the Bayesian estimates which help the physician to compute the effects of the evidence in increasing the probability that whether or not a particular patient has a disease by using the prior information. The current study emphasis on the development of the Bayesian estimates of the Lomax Exponential distribution with four loss function under two non-informative priors. The four loss functions are the Square error loss function, Quadratic loss function, Weighted loss function, and the precautionary error loss function. The Lomax Exponential distribution is recently developed by Ijaz et al. (2019).

Let a positive continuous random variable X follows a Lomax Exponential distribution with parameter $a, b > 0$.

Then the Cumulative distribution function is defined as:

$$F(x) = 1 - \left[ 1 + \left( \frac{xe^{xp(x)}}{b} \right) \right]^{-a} , \quad a, b > 0, \quad x > 0$$

(1)

The corresponding probability density function is given as:

$$f(y) = \frac{a}{b} (x+1)e^{xp(x)} \left[ 1 + \left( \frac{xe^{xp(x)}}{b} \right) \right]^{-a+1}$$

(2)

There exist two approaches to deal with a real problem using Statistics. The first approach is a classical in which the parameters are considered a fixed quantity. However, in a real situation, a lot of problems exist particularly in
modeling the life time data where the parameters are to be considered a variable (Bayesian approach) (Ibrahim et al., 2001; Martz and Waller, 1982; Singpurwalla, 2006). Hence the Bayesian approach is also an important phase of statistics.

The Bayesian analysis has attracted researchers to analyze the data under Bayesian paradigm rather than the classical approach. For example, Hasan and Baizid (2017) discussed the Bayesian analysis of the of Exponential distribution, Canavos and Taokas (1973) presented the Bayesian analysis of the Weibull distribution, Guure et al. (2012) explored the Bayesian estimation of Weibull distribution under the Extension of Jeffrey's' prior. Okasha (2014) presented the Bayesian estimation of the Lomax distribution under type-II censored data. Son and Oh (2006) worked on the Bayesian estimation of the two-parameter Gamma distribution. Raqab and Madi (2005) emphasized the Bayesian inference for the generalized exponential distribution. Jan and Ahmad (2017) presented the Bayesian analysis of the inverse Lomax distribution.

The main aim of the study is to develop the Bayesian estimates for a shape parameter of the Lomax Exponential distribution so that to make a statistical comparison between the Maximum likelihood and Bayesian estimates under a Uniform and Jeffery priors. Also to delineate the behaviors of these estimators, we have increased the sample size (n), and to detailed discuss the Bayesian estimates of a shape parameter under a Uniform and Jeffery priors with four different loss functions. The estimators are computed and compared by means of Mean Square error for both the simulated and applied data.

**MATERIALS AND METHODS**

**Prior and posterior distribution**

To estimate the Bayes estimator, we need to specify the prior probability distribution. In Bayesian analysis, there is no way to select the appropriate prior information to perform the analysis. There are two ways to select the prior information about the model under study. These are informative priors and non-informative priors. In practice, one prefers to use the informative priors if they have sufficient adequate information about the parameters of a model. In this study, we have considered two non-informative priors; the Uniform and Jeffery priors.

In general, the posterior probability distribution of a shape parameter by using the Bayes theorem is given as:

\[ f(a/x_i) = \frac{\pi(a) \prod_{i=1}^{n} f(x_i/a)\pi(a)}{\int_{a} f(x_i/a) f(a)da} \]

where \( \pi(a) \) is the prior probability function and \( n \sum_{i=1}^{n} f(x_i/a) \) is the likelihood function.

**Prior and posterior distribution under uniform prior**

The prior distribution for the Lomax Exponential distribution under uniform prior is given as:

\[ \pi(a) \propto 1 \]

Then the posterior distribution of the Lomax Exponential distribution is defined as:

\[ f(a/x_i) = \frac{n \prod_{i=1}^{n} f(x_i/a)\pi(a)}{\int_{a} f(x_i/a) f(a)da} \]  \hspace{1cm} (3)

where \( \prod_{i=1}^{n} f(x_i/a) \) is the maximum likelihood function of the Lomax Exponential distribution and is defined as:

\[ n \prod_{i=1}^{n} f(x_i/a) = \left( \frac{a}{b} \right)^n \exp\left[ \sum log\left[ (x_i + 1) \exp(x_i) \right] - (a + 1) \sum log \left[ 1 + \left( \frac{x_i \exp(x_i)}{b} \right) \right] \right] \]

By applying the log-likelihood and equate the result to zero, we finally obtained:

\[ \sum \log \left( 1 + \frac{y_i \exp(y_i)}{b} \right) = \frac{B}{n} \]

By putting the maximum likelihood function in (3), we obtained:

\[ f(a/x_i) = \frac{\left( \frac{a}{b} \right)^n \exp\left[ \sum log\left[ (x_i + 1) \exp(x_i) \right] - \left( a + 1 \right) \sum log \left[ 1 + \left( \frac{x_i \exp(x_i)}{b} \right) \right] \right]}{\int \left( \frac{a}{b} \right)^n \exp\left[ \sum log\left[ (x_i + 1) \exp(x_i) \right] - \left( a + 1 \right) \sum log \left[ 1 + \left( \frac{x_i \exp(x_i)}{b} \right) \right] \right]da} \]

By solving the above expression, we finally obtained:

\[ f(a/x_i) = \frac{B^{n+1}}{\Gamma(n+1)} a^n \exp(-aB), \quad a > 0 \]  \hspace{1cm} (4)
This implies that \( f(a/x_i) \sim \text{Gamma}(n+1, B) \).

where, \( B = \sum_{i=1}^{n} \log \left( 1 + \frac{x_i \exp(x_i)}{b} \right) \).

Prior and posterior distribution under Jeffery prior

The prior distribution for the Lomax Exponential distribution under Jeffery prior is obtained as:

\[
\pi(a) = \left[ -E \frac{\partial^2 \log L}{\partial a^2} \right] = \left[ -E \left( \frac{\partial^2 \log L(x_i, a)}{\partial a^2} \right) \right] = \left[ \frac{n}{a^2} \right]^{1/2}
\]

The log-likelihood function of the Lomax Exponential distribution is given as:

\[
L(x; a, b) = n \log \left( \frac{a}{b} \right) + \sum \log(y + 1) + \sum x_i - (a + 1) \sum \log \left( 1 + \left( \frac{x \exp(x_i)}{b} \right) \right)
\]

Differentiating with respect to the parameter \( a \), we have:

\[
\frac{\partial \log L}{\partial a} = \frac{n}{a} - \sum \log \left( 1 + \left( \frac{x \exp(x_i)}{b} \right) \right)
\]

Again, differentiating with respect to the parameter \( a \), we have:

\[
\frac{\partial^2 \log L(x_i, a)}{\partial a^2} = -\left( \frac{n}{a^2} \right)^{1/2}
\]

Taking expectation and the square root of the above equation, we obtained:

\[
\left[ -E \left( \frac{\partial^2 \log L(x_i, a)}{\partial a^2} \right) \right]^{1/2} = \left( \frac{n}{a^2} \right)^{1/2}
\]

\[
\left[ -E \left( \frac{\partial^2 \log L(x, a)}{\partial a^2} \right) \right]^{1/2} = \frac{n}{a}
\]

Finally, we obtained the Jeffery prior as:

\[
\pi(a) = f(a) \propto a^{-1}
\]

Finally, the posterior probability distribution is defined as:

\[
f(a/x) = \frac{\left( \frac{a}{b} \right)^n \exp \left( \sum \log[(x_i + 1) \exp(x_i)] - (a + 1) \sum \log \left( 1 + \left( \frac{x \exp(x_i)}{b} \right) \right) \right)}{\Gamma(n) a^{n-1} \exp(-aB)}
\]

The posterior distribution takes the following form:

\[
f(a/x) = \frac{B^n}{\Gamma(n)} a^{n-1} \exp(-aB), a > 0
\]

This implies that \( f(a/x) \sim \text{Gamma}(n, B) \), where,

\[
B = \sum_{i=1}^{n} \log \left( 1 + \frac{x \exp(x_i)}{b} \right)
\]

Bayesian estimators under uniform prior, using different loss functions

Here, we derived the Bayes estimators of the shape parameter \( a \) under a uniform prior by using four loss functions. The detailed discussion is given hereafter.

Squared error loss function

The Bayes estimator of \( a \) under squared error loss (Singh et al., 2011) function is defined as:

\[
L(\hat{a}, a) = (\hat{a} - a)^2
\]

Using the above loss function, the Bayes estimator of \( a \) is defined by solving the equation:

\[
\frac{\partial}{\partial \hat{a}} \left( (\hat{a} - a)^2 \right) = 0
\]

\[
\hat{a} = \frac{n}{B} (\frac{n+1}{B})
\]
Finally, we determined the result for \( \hat{a} \):

\[
\hat{a}_{SELF} = \frac{(n+1)}{B}
\]  

(10)

**Quadratic loss function**

The Bayes estimator of \( a \) under quadratic loss function (Azam and Ahmad, 2014) is defined as:

\[
L(\hat{a},a) = \left( \frac{\hat{a} - a}{a} \right)^2
\]

The estimator \( \hat{a} \) is defined by solving the equation:

\[
\frac{\partial}{\partial \hat{a}} \int L(\hat{a},a)f\left(\frac{a}{x}\right)da = 0
\]

\[
\frac{\partial}{\partial \hat{a}} \int \left( \frac{\hat{a} - a}{a} \right)^2 f\left(\frac{a}{x}\right)da = 0
\]

\[
\frac{B^{n+1}}{\Gamma(n+1)} \hat{a} \int a^{n-2} \exp(-aB)da - \int a^{n+1} \exp(-aB)da = 0
\]

\[
\hat{a} \frac{\Gamma(n-1)}{B^{n-1}} - \frac{\Gamma(n)}{B^n} = 0
\]

Finally, we determined the following result:

\[
\hat{a}_{WELF} = \frac{(n)}{B}
\]  

(12)

**Precautionary loss function**

The Bayes estimator of \( a \) under precautionary loss function (Azam and Ahmad, 2014a) is defined as:

\[
L(\hat{a},a) = \left( \frac{\hat{a} - a}{a} \right)^2
\]

The Bayes estimator \( \hat{a} \) under precautionary loss function is defined by the equation:

\[
\frac{\partial}{\partial \hat{a}} \int L(\hat{a},a)f\left(\frac{a}{x}\right)da = 0
\]

\[
\frac{\partial}{\partial \hat{a}} \int \left( \frac{\hat{a} - a}{a} \right)^2 f\left(\frac{a}{x}\right)da = 0
\]

\[
\frac{B^{n+1}}{\Gamma(n+1)} \hat{a} \int \frac{\sum x_i^n}{\hat{a} \Gamma n} e^{-a \sum x_i} da = 0
\]

\[
\frac{\partial}{\partial \hat{a}} \int \frac{\hat{a}^2 - a^2}{\hat{a}^2} \frac{\sum x_i^n}{\Gamma n} a^{n+1} e^{-a \sum x_i} da = 0
\]

\[
\frac{B^{n+1}}{\Gamma(n+1)} \int a^n \exp(-aB)da - \frac{1}{\hat{a}^2} \int a^{n+3} \exp(-aB)da = 0
\]
\[
\frac{\Gamma(n+1)}{B^{n+1}} - \frac{1}{a^2} \frac{\Gamma(n+3)}{B^{n+3}} = 0
\]

Hence, we determined the result:

\[
\hat{a}_{PLF} = \frac{\sqrt{(n+1)(n+2)}}{B}
\]  \hspace{1cm} \text{(13)}

Bayesian estimators under Jeffery prior, using different loss functions

Here, we derived Bayes estimators of the parameter \( a \) under Jeffery prior by using various loss functions. The detailed discussion is given hereafter.

**Squared error loss function**

The Bayes estimator of \( a \) under squared error loss (Singh et al., 2011) function is defined as:

\[
L(\hat{a}, a) = (\hat{a} - a)^2
\]

Using the above loss function, the Bayes estimator of \( a \) is defined by solving the equation:

\[
\frac{\partial}{\partial \hat{a}} \int (\hat{a} - a)^2 f(a/x) da = 0
\]

\[
\int \hat{a} f(a/x) da - a \int f(a/x) da = 0
\]

\[
\frac{B^n}{\Gamma(n)} \left[ \hat{a} \int a^{n-1} \exp(-aB) da - \int a^{n-1} \exp(-aB) da \right] = 0
\]

Finally, we determined the following result:

\[
\hat{a}_{QELF} = \frac{(n-2)}{B}.
\]  \hspace{1cm} \text{(15)}

**Weighted square loss function**

The Bayes estimator of \( a \) under weighted square loss function (Zhang, 2019) is defined as:

\[
L(\hat{a}, a) = \frac{(\hat{a} - a)^2}{a}
\]

The estimator \( \hat{a} \) is defined by solving the equation:

\[
\frac{\partial}{\partial \hat{a}} \int L(\hat{a}, a) da = 0
\]

\[
\int \hat{a} f(a/x) da - a \int f(a/x) da = 0
\]

Finally, we determined the result for \( \hat{a} \):

\[
\hat{a}_{SELF} = \frac{n}{B}.
\]  \hspace{1cm} \text{(14)}

**Quadratic loss function**

The Bayes estimator of \( a \) under quadratic loss function (Azam and Ahmad, 2014) is defined as:

\[
L(\hat{a}, a) = \left(\frac{\hat{a} - a}{a}\right)^2
\]

The estimator \( \hat{a} \) is defined by solving the equation:

\[
\frac{\partial}{\partial \hat{a}} \int L(\hat{a}, a) da = 0
\]

\[
\int \frac{\hat{a}}{a} f(a/x) da - a \int f(a/x) da = 0
\]
\[ \frac{B^n}{\Gamma(n)} \left\{ \frac{\hat{a} - a}{\hat{a}} \int a^{n-1} \exp(-aB) \, da - \int a^{n-1} \exp(-aB) \, da \right\} = 0 \]

Finally, we get the following result:

\[ \hat{a}_{WELF} = \frac{(n - 1)}{B} \]  \hspace{1cm} (17)

**Precautionary loss function**

The Bayes estimator of \( a \) under the precautionary loss function (Azam and Ahmad, 2014a) is defined as:

\[ L(\hat{a}, a) = \left( \frac{\hat{a} - a}{\hat{a}} \right)^2 \]

The Bayes estimator \( \hat{a} \) under the precautionary loss function is defined by the equation:

\[ \frac{\partial}{\partial \hat{a}} \int L(\hat{a}, a) \, da = 0 \]

\[ \frac{\hat{a}^2}{\Gamma n} \int (\hat{a} - a)^2 a^{n-1} \left( \sum x_i \right)^n e^{-\hat{a} \sum x_i} \, da = 0 \]

\[ \frac{\hat{a}^2}{\Gamma n} \int \frac{\hat{a}^2 - a^2}{\hat{a}^2} \left( \sum x_i \right)^n \frac{a^{n-1} e^{-a \sum x_i}}{\Gamma n} \, da = 0 \]

\[ \frac{B^n}{\Gamma(n)} \left\{ \int a^{n-1} \exp(-aB) \, da - \frac{1}{\hat{a}^2} \int a^{n+2} \exp(-aB) \, da \right\} = 0 \]

\[ \frac{\Gamma(n) - 1}{\hat{a}^2} \frac{\Gamma(n + 2)}{B^{n+2}} = 0 \]

Hence, we determined the result:

\[ \hat{a}_{PLF} = \sqrt{n(n + 1)} \]  \hspace{1cm} (18)

**Quantile function**

To perform a simulation study, we have to generate random data from the Lomax Exponential distribution. This can be done by using the quantile function of the Lomax Exponential distribution. Whereas the quantile function is the real solution to the inverse cumulative distribution function of the Lomax Exponential distribution. Let \( X \) be a Lomax Exponential random variable, then, the quantile function is defined as:

\[ F(x) = u \]

\[ 1 - \left[ 1 + \left( \frac{x \exp(x)}{b} \right)^{-a} \right] = u \]  \hspace{1cm} (19)

where \( u \) is uniformly distributed over the interval \([0,1]\).

Now, solving for \( x \), we finally obtained the following result:

\[ x = W \left( \frac{b}{(1 - u)^{1/a} - b} \right), \]  \hspace{1cm} (20)

where \( W(\cdot) \) is the product log function.

**RESULTS AND DISCUSSION**

A Monte Carlo simulation method is adopted while generating sample data with 6000 replications from the Lomax Exponential distribution under uniform and Jeffery priors. A short algorithm for the simulation study is as follows:

i. Generate samples from the Lomax Exponential distribution by using the quantile function \( x = W \left( \frac{b}{(1 - u)^{1/a} - b} \right) \), where \( U \) is the standard uniform random variable.

ii. Obtain Bayes estimator of various loss functions under a uniform and Jeffery prior.

iii. The above steps are repeated 6000 times for each sample size and obtained the estimated values of Bayes estimators and their MSE.

We computed the estimated values of \( \hat{a}_{SLF}, \hat{a}_{QLF}, \hat{a}_{WELF}, \hat{a}_{PLF} \) under uniform and Jeffery priors. For the purpose of comparison, we considered the mean squared error (MSE) of the estimator which is defined as:
Table 1: Estimated value and MSE of a shape parameter under a uniform Prior, when a=0.5 and b=2.5.

<table>
<thead>
<tr>
<th>n</th>
<th>Criteria</th>
<th>MLE</th>
<th>BSE</th>
<th>BQEL</th>
<th>BWEL</th>
<th>BPEL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimated value</td>
<td>0.6256811</td>
<td>0.7380114</td>
<td>0.5032686</td>
<td>0.6256811</td>
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<tr>
<td></td>
<td>MSE</td>
<td>0.1442188</td>
<td>0.2528383</td>
<td>0.0845336</td>
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<td>0.3197041</td>
</tr>
<tr>
<td>10</td>
<td>Estimated value</td>
<td>0.5553952</td>
<td>0.6062857</td>
<td>0.5015279</td>
<td>0.5553952</td>
<td>0.6405076</td>
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<tr>
<td></td>
<td>MSE</td>
<td>0.0412943</td>
<td>0.05382705</td>
<td>0.03204627</td>
<td>0.0412943</td>
<td>0.0714784</td>
</tr>
<tr>
<td>15</td>
<td>Estimated value</td>
<td>0.5373733</td>
<td>0.5672753</td>
<td>0.5001493</td>
<td>0.5373733</td>
<td>0.5878884</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.02322292</td>
<td>0.0292292</td>
<td>0.01849015</td>
<td>0.02322292</td>
<td>0.03418804</td>
</tr>
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<td>20</td>
<td>Estimated value</td>
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<td>0.547001</td>
<td>0.5014116</td>
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<td>0.01869546</td>
<td>0.0139216</td>
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</tr>
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<td>25</td>
<td>Estimated value</td>
<td>0.5188773</td>
<td>0.53797</td>
<td>0.5029713</td>
<td>0.5188773</td>
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<tr>
<td></td>
<td>MSE</td>
<td>0.01168387</td>
<td>0.01400082</td>
<td>0.01144029</td>
<td>0.01168387</td>
<td>0.01615689</td>
</tr>
<tr>
<td>30</td>
<td>Estimated value</td>
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<td>0.5337677</td>
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<td>0.5152947</td>
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<tr>
<td></td>
<td>MSE</td>
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<td>0.008845529</td>
<td>0.009390352</td>
<td>0.011993</td>
</tr>
</tbody>
</table>

Figure 1: Graph of MSE for Different estimators when a=0.5 and b=2.5.

\[ \text{MSE}(\hat{a}) = E((\hat{a} - a)^2) = \text{var}(\hat{a}) + \left[\text{Bias}(\hat{a})\right]^2 \]

We assumed the values of the parameter “a” and “b” to test these estimators’ behaviors in both classical and non-classical methods by increasing the sample of size n. The results of the estimated values of a shape parameter and their MSE for different loss functions are given in Tables 1 to 2.

Table 1 clearly demonstrates that the Bayes estimator for the Quadratic loss function under a uniform prior provides a better result as compared with other loss functions. The MLE estimator and BWEL provide the same results. Moreover, the table clearly indicates that the Bayes estimator under BQEL and BWEL are very close to each other and as we increase the sample of size n, these two estimators rapidly become identical. Furthermore, the estimator under BSE and BPEL are very close to each other, for the large sample size, these two estimators become identical. Figure 1 shows that the estimator BQEL performs better as compared with others and become rapidly identical to BWEL for the large values of n.
Table 2: Estimated value and MSE of a shape parameter under Jeffery Prior, when \( a=0.5 \) and \( b=2.5 \).

<table>
<thead>
<tr>
<th>n</th>
<th>Criteria</th>
<th>MLE</th>
<th>BSEL</th>
<th>BQEL</th>
<th>BWEL</th>
<th>BPEL</th>
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<tr>
<td>5</td>
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<td>0.6280205</td>
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<td>MSE</td>
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<td>0.1549811</td>
<td>0.06065319</td>
<td>0.07666999</td>
<td>0.1844328</td>
</tr>
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<td>Estimated value</td>
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<td>0.5531069</td>
<td>0.4463</td>
<td>0.4986207</td>
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<tr>
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<td>MSE</td>
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<td>0.0374295</td>
<td>0.02792077</td>
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<td>0.04834073</td>
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<td>Estimated value</td>
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<td>0.5346654</td>
<td>0.4642361</td>
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<td></td>
<td>MSE</td>
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<td>0.0223632</td>
<td>0.01762017</td>
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<td>0.0159023</td>
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<td>25</td>
<td>Estimated value</td>
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<td>0.5203168</td>
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<td></td>
<td>MSE</td>
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<td>0.0123723</td>
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<td>30</td>
<td>Estimated value</td>
<td>0.5180645</td>
<td>0.5180645</td>
<td>0.4831925</td>
<td>0.4994207</td>
<td>0.5251306</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0101960</td>
<td>0.0101960</td>
<td>0.00856639</td>
<td>0.00878592</td>
<td>0.01039535</td>
</tr>
</tbody>
</table>

Figure 2: Graph of MSE for different estimators when \( a=0.5 \) and \( b=2.5 \).

Table 2 also shows similar result as in Table 1. The Bayes estimator for the Quadratic loss function under Jeffery prior provides a better result as compared with other loss functions. The results of MLE and BSEL are the same. The table clearly indicates that the Bayes estimator under BQEL and BWEL are very close to each other and as we increase the sample of size \( n \), these two estimators rapidly become identical. Furthermore, the estimator under BSE and BPEL are very close to each other, for the large sample size, and these two estimators become identical. Figure 2 shows the behavior of different estimators towards the sample of size \( n \). It can be seen that the lines of BQEL and BWEL are close to each other and become identical as the sample size increases. However, the estimators BSEL and BPEL are similar to each other.

**Application**

This section presents the real-life application of the Bayesian estimators under a Uniform and Jeffery priors by using four loss functions. The data set describes the
Table 3: Estimated value and MSE of a shape parameter under Uniform Prior, when a=0.1 and b=1.5.

<table>
<thead>
<tr>
<th>n</th>
<th>Criteria</th>
<th>MLE</th>
<th>BSE</th>
<th>BQEL</th>
<th>BWEL</th>
<th>BPEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Estimated value</td>
<td>0.1284485</td>
<td>0.152872</td>
<td>0.1030451</td>
<td>0.1284485</td>
<td>0.1666078</td>
</tr>
<tr>
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<td>0.002378757</td>
<td>0.004692761</td>
<td>0.001057165</td>
<td>0.002378757</td>
<td>0.006990193</td>
</tr>
<tr>
<td>10</td>
<td>Estimated value</td>
<td>0.1230886</td>
<td>0.1353034</td>
<td>0.1108587</td>
<td>0.1230886</td>
<td>0.1412985</td>
</tr>
<tr>
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<td>MSE</td>
<td>0.0009401202</td>
<td>0.001807571</td>
<td>0.0004783025</td>
<td>0.0009401202</td>
<td>0.002268774</td>
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<tr>
<td>15</td>
<td>Estimated value</td>
<td>0.121606</td>
<td>0.1304037</td>
<td>0.1136113</td>
<td>0.121606</td>
<td>0.1339462</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0006604507</td>
<td>0.001151087</td>
<td>0.0003580039</td>
<td>0.0006604507</td>
<td>0.001394005</td>
</tr>
<tr>
<td>20</td>
<td>Estimated value</td>
<td>0.1205859</td>
<td>0.1264412</td>
<td>0.1147477</td>
<td>0.1205859</td>
<td>0.1297814</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0005212664</td>
<td>0.0008026753</td>
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<td>0.0009980501</td>
</tr>
<tr>
<td>25</td>
<td>Estimated value</td>
<td>0.1202134</td>
<td>0.1250995</td>
<td>0.1154971</td>
<td>0.1202134</td>
<td>0.1273932</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0004473539</td>
<td>0.0006721924</td>
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<td>0.0004473539</td>
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</tr>
<tr>
<td>30</td>
<td>Estimated value</td>
<td>0.1200376</td>
<td>0.1239542</td>
<td>0.1160049</td>
<td>0.1200376</td>
<td>0.1259822</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0004071126</td>
<td>0.0005798273</td>
<td>0.0002614134</td>
<td>0.0004071126</td>
<td>0.0006813319</td>
</tr>
</tbody>
</table>

Table 4: Estimated value and MSE of a shape parameter under Jeffery Prior, when a=0.1 and b=1.5.

<table>
<thead>
<tr>
<th>n</th>
<th>Criteria</th>
<th>MLE</th>
<th>BSE</th>
<th>BQEL</th>
<th>BWEL</th>
<th>BPEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Estimated value</td>
<td>0.1249413</td>
<td>0.1249413</td>
<td>0.07510783</td>
<td>0.09925074</td>
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<tr>
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<td>MSE</td>
<td>0.005823846</td>
<td>0.005823846</td>
<td>0.002398796</td>
<td>0.002993371</td>
<td>0.007774942</td>
</tr>
<tr>
<td>10</td>
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<td>0.1113237</td>
<td>0.1113237</td>
<td>0.08859174</td>
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</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.00167037</td>
<td>0.00167037</td>
<td>0.001133659</td>
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<td>0.001923252</td>
</tr>
<tr>
<td>15</td>
<td>Estimated value</td>
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<td>0.1078045</td>
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<td>0.09939381</td>
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<tr>
<td></td>
<td>MSE</td>
<td>0.0009822888</td>
<td>0.0009822888</td>
<td>0.0007311753</td>
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<tr>
<td>20</td>
<td>Estimated value</td>
<td>0.10538</td>
<td>0.10538</td>
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<td>0.1008003</td>
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<td></td>
<td>MSE</td>
<td>0.0006649268</td>
<td>0.0006649268</td>
<td>0.0005484903</td>
<td>0.0005795853</td>
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</tr>
<tr>
<td>25</td>
<td>Estimated value</td>
<td>0.1040255</td>
<td>0.1040255</td>
<td>0.09616459</td>
<td>0.1000298</td>
<td>0.1058271</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>0.0004940745</td>
<td>0.0004940745</td>
<td>0.0004267834</td>
<td>0.0004329605</td>
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<td>30</td>
<td>Estimated value</td>
<td>0.1039575</td>
<td>0.1039575</td>
<td>0.09666163</td>
<td>0.1000574</td>
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<tr>
<td></td>
<td>MSE</td>
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<td>0.0003988354</td>
<td>0.0003553191</td>
<td>0.0003572925</td>
<td>0.0004232773</td>
</tr>
</tbody>
</table>

Duration of hospitalization (on daily basis) in a neurosurgical service of a research hospital in Turkey for 31 male patients. The data set has been cited by Gunay and Yilmaz (2018). The data set values are as follows:

1, 1, 1, 1, 1, 2, 3, 4, 4, 4, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 9, 10, 11, 11, 11, 12, 12, 13, 14, 17.

Also, we assumed the values of parameter "a" and "b" so as to study the behaviors of these estimators under classical and non-classical approaches by increasing the sample of size n. The Estimated and MSE values of a shape parameter under a Uniform and Jeffery priors are respectively given in Tables 3 and 4.

Figure 3 shows the histogram and theoretical density of the Duration of hospitalization in a neurosurgical service. The graph clearly indicates that the data is stretched out to the right and follows a positively skewed distribution. By considering a real data set, a similar passion is seen in Table 3 as found in Table 1 for the Bayesian estimates of a
shape parameter under a uniform prior. Table 3 shows that the Bayes estimator for the Quadratic loss function under a uniform prior provides a better result as compared with other loss functions. The table clearly indicates that the Bayes estimator under BQEL and BWEL are very close to each other and as we increase the sample of size n, these two estimators rapidly become identical. Furthermore, the estimator under BSE and BPEL is very close to each other, for the large sample size, these two estimators become identical. Figure 4 shows that the estimator BQEL performs better as compared with others and becomes rapidly identical to BWEL.

By considering a real data set, a similar passion is seen in Table 4 as found in Table 2 for the Bayesian estimates of a shape parameter under a uniform prior. Table 3 shows that the Bayes estimator for the Quadratic loss function under a uniform prior provides a better result as compared with other loss functions. The table clearly indicates that the
Bayes estimator under BQEL and BWEL are very close to each other and as we increase the sample of size $n$, these two estimators rapidly become identical. Furthermore, the estimator under BSE and BPEL is very close to each other, for the large sample size, these two estimators become identical. Figure 5 shows that the estimator BQEL performs better as compared with others and becomes rapidly identical to BWEL.

**Conclusion**

In this study, we derived the Bayesian estimators of the shape parameter of the Lomax Exponential distribution under a Uniform and Jeffery priors. Bayes estimates and their MSE are computed by using four loss functions under the two prior distributions. The four loss functions are Square error loss function (SEL), Quadratic error loss function (QEL), Weighted error loss function (WEL), and Precautionary error loss function (PEL). The performance of these estimators has been studied with both a simulation as well as a real data set. The performance of different estimators is judged by using their Mean Square error (MSE). Based on the numerical illustration and graphical representation, it is evident that to estimate the shape parameter of the Lomax Exponential distribution, the use of Quadratic loss function in both under a Uniform and Jeffery priors will provide better results rather than others irrespective of the values of parameter “a” and “b”. Moreover, as we increase the sample of size $n$, the results of all estimators become approximately identical.

**REFERENCES**


